

ELASTICA AS A DYNAMICAL SYSTEM

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ABSTRACT. The elastica is a curve in \mathbf{R}^3 that is stationary under variations of the integral of the square of the curvature. Elastica are viewed as a dynamical system that arises from the second order calculus of variations, and its quantization is discussed.

1. INTRODUCTION

Ever since the beginning of the calculus of variations, second order problems such as the classical problem of the elastica have been considered. The peculiar situation that distinguishes most of the interesting examples in second order problems from the more familiar first order theory is that they are parameter independent, and so the theory of such problems has a somewhat distinctive tone from that of the more familiar first order theory. A comprehensive review of this theory, as it was understood up until the 1960s, may be found in the monograph by Grässer [7].¹ By way of contrast, this paper emphasizes the relation of the variational problem to the geometry of the corresponding Cartan form and its interplay with symmetry, conservation laws, the Noether theorems and the associated first order canonical formalism. The problem of elastica is then re-examined in light of this discussion.

The rationale for this paper is to give a biased view of that portion of the theory of second order variational problems that could reasonably be expected to be useful for understanding the common behaviour of several geometric functionals on curves. Good examples to keep in mind while reading this paper (these are the three main examples that motivated our study) are the elastica, the shape of a real Möbius band in terms of the geometry of the central geodesic [20], and the curve of least friction.

For reasons not entirely clear to us, the geometric theory of higher order variational problems seems to have developed in a manner largely detached from the needs and concerns of concrete problems. This is a startling contrast to recent developments in geometric mechanics and their understanding of stability, bifurcation, numerical schemes, the incorporation of nonholonomic constraints, etc. The consequences of this are at least two-fold: first, it leads to a palpable sense of dread² when faced with trying to look up a formulation of some part of the theory that will cleanly explain how to compute something obvious, and second, a real disconnect between the theoretical insights and the actual computational methods. This disconnect is vividly illustrated in the problem of elastica.

¹This monograph is especially noteworthy for its comprehensive bibliography.

²This may be reduced to mere frustration by those less ignorant than the authors.

Planar elastica (the equilibrium shape of a linearly elastic thin wire) were considered at least as early as 1694 by James Bernoulli.³ However, it was not until about 1742 that Daniel Bernoulli convinced Euler to solve the problem by using the isoperimetric method (the old name for the calculus of variations before Lagrange.) From a variational point of view, the elastica is idealized as a curve that minimizes the integral over its length of the square of the curvature (that is, minimize $\int \kappa^2 ds$), and is thus naturally treated as a second order problem in the calculus of variations. Exhaustive results were then published by Euler in 1744 [5]. Since Euler's results were so comprehensive, it is not surprising that the study of elastica remained somewhat dormant until taken up again by Max Born in his thesis [2]. More recently a striking result was obtained in 1984 by Langer and Singer [12] when they demonstrated the existence of closed elastica that were torus knots. Their proof was noteworthy because they eschewed the usual variational machinery and employed clever *ad hoc* geometric arguments such as an adapted cylindrical coordinate system to aid their integration. In fact, a significant motivation for this paper was to see to what extent their results could be understood by a more pedestrian use of the second order calculus of variations that looked more like just 'turning the crank' on the variational machine, and thus had the comfort of familiarity of technique.

Some features of the elastica problem instantly spring to mind in the modern geometrically oriented reader. The first is that the problem is manifestly invariant by the action of the Euclidean group. The second is that it would be very nice to have a theory that explained how to reduce the symmetry using the concomitant conservation laws that Emmy Noether taught us are in the problem, and then wind up with some form of reduced Euler-Lagrange equations. Assuming we can solve these reduced equations, and hence know the curvature and torsion of the elastic curve, we would expect a good theory to show us methods to determine the shape of our curve that go beyond a trite referral to the fundamental theorem of curves stating that the curvature and torsion of the curve determine it up to a Euclidean motion. Given all this, what we actually find when we look at the published work on elastica (such as [12] or [3]) is that it proceeds somewhat differently. In particular, almost none of the actual computations seem to follow any method that resembled the current theory. There are good reasons for this, and it is not due to ignorance of those geometers but a reflection that the theory at that time was presented in such a way as to simply be unhelpful, and unable to easily identify the geometric meaning of some of their calculations. This is the best explanation we have of the situation at the time and why it was still necessary a decade after [12] appeared for Foltinek (see [6]) to write a paper demonstrating the integration constants in elastica in terms of the conserved Noetherian momenta.

The plan of this paper is to first discuss the Euler-Lagrange equations. This section will serve to fix notation and some basic notions. Next, we discuss the Cartan form as the geometrization of the variational problem using the structure of the jet bundle. This is followed by discussion of parametrization invariance and

³See the delightful discussion by Levien in [14] or [13].

the Hamiltonian formalism. Elastica are then studied from this point of view. The problem is then recast as a constrained Hamiltonian system and the reparametrization group and its reduction are examined. The paper concludes with the geometric quantization, and discusses the quantum representations of the groups $SE(3)$ and $\text{Diff}_+ \mathbf{R}$ as well as the quantum implementation of constraints.

2. SECOND ORDER VARIATIONAL PROBLEMS

2.1. Variation of the action integral. A curve $[t_0, t_1] \rightarrow \mathbf{R}^n : t \mapsto x(t)$ can be uniquely described by the corresponding section

$$\sigma : [t_0, t_1] \rightarrow [t_0, t_1] \times \mathbf{R}^n : t \mapsto (t, x(t)). \quad (1)$$

We consider $Q = \mathbf{R} \times \mathbf{R}^n$ as a bundle over \mathbf{R} with typical fibre \mathbf{R}^n . For each integer $k \geq 0$, we denote by J^k the k -th jet of sections of Q . Furthermore, interpret the section σ given in (1) as a local section of Q and denote by $j^k \sigma : [t_0, t_1] \rightarrow J^k$ the k -jet extension of σ . This paper considers variational problems defined by second order Lagrangians. For a Lagrangian $L : J^2 \rightarrow \mathbf{R} : (t, x, \dot{x}, \ddot{x}) \mapsto L(t, x, \dot{x}, \ddot{x})$, the corresponding action integral is

$$A(\sigma) = \int_{t_0}^{t_1} (L \circ j^2 \sigma) dt = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t)) dt.$$

A variation of a section (without variation of time) $\sigma \mapsto \sigma + \delta \sigma : t \mapsto x(t) + \delta x(t)$ extends to the second jets as

$$j^2 \sigma \mapsto j^2 \sigma + \delta j^2 \sigma : t \mapsto (x(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), \ddot{x}(t) + \delta \ddot{x}(t)),$$

where

$$\delta \dot{x}(t) = \frac{d}{dt} \delta x(t) \quad \text{and} \quad \delta \ddot{x}(t) = \frac{d}{dt} \delta \dot{x}(t) = \frac{d^2}{dt^2} \delta x(t).$$

Then, integrating by parts twice, it follows that the action varies as

$$\begin{aligned} \delta A(\sigma) &= \int_{t_0}^{t_1} \delta L(t, x(t), \dot{x}(t), \ddot{x}(t)) dt \\ &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x}(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(t) + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}}(t) \right\} \delta x dt + \\ &\quad + \left(\frac{\partial L}{\partial \dot{x}}(t) - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}}(t) \right) \delta x \Big|_{t_0}^{t_1} + \frac{\partial L}{\partial \ddot{x}}(t) \delta \dot{x} \Big|_{t_0}^{t_1}. \end{aligned}$$

It follows from the fundamental lemma of the calculus of variations that

CONCLUSION 2.1. The action integral

$$A(\sigma) = \int_{t_0}^{t_1} (L \circ j^2 \sigma) dt = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t)) dt$$

is stationary with respect to all variations $\sigma \mapsto \sigma + \delta \sigma : t \mapsto x(t) + \delta x(t)$, such that δx and $\delta \dot{x}$ vanish on the boundary, if and only if the section σ satisfies the

Euler-Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} = 0. \quad (2)$$

Consider now the boundary terms in the variation. The partial derivative $\frac{\partial L}{\partial \ddot{x}}$ is a map from J^2 to \mathbf{R}^n , and

$$\frac{\partial L}{\partial \ddot{x}}(t) \delta \dot{x} = \left\langle \frac{\partial L}{\partial \ddot{x}}(t, x(t), \dot{x}(t), \ddot{x}(t)), \delta \dot{x}(t) \right\rangle,$$

where the angle bracket denotes the Euclidean scalar product in \mathbf{R}^n . However,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}}(t) \right) = \left(\frac{\partial}{\partial t} \frac{\partial L}{\partial \ddot{x}} + \dot{x} \frac{\partial}{\partial x} \frac{\partial L}{\partial \ddot{x}} + \ddot{x} \frac{\partial}{\partial \dot{x}} \frac{\partial L}{\partial \ddot{x}} + \ddot{\ddot{x}} \frac{\partial}{\partial \ddot{x}} \frac{\partial L}{\partial \ddot{x}} \right)$$

depends on the third derivative $\ddot{\ddot{x}}$ of the section σ , and hence, $\frac{d}{dt} \frac{\partial L}{\partial \ddot{x}}$ can be interpreted as a map from J^3 to \mathbf{R}^n . Using the projection map

$$\pi_{32} : J^3 \rightarrow J^2 : (t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \mapsto (t, x, \dot{x}, \ddot{x}),$$

define Ostrogradski's momenta by

$$\begin{aligned} p_{\dot{x}} &= \pi_{32}^* \frac{\partial L}{\partial \ddot{x}}, \\ p_x &= \pi_{32}^* \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}}, \end{aligned}$$

and interpret them as maps from J^3 to \mathbf{R}^n . In the following, in order to simplify the notation, the pull-back sign is omitted and an overdot is used to denote the derivative with respect to t . This leads to the usual expressions

$$\begin{aligned} p_{\dot{x}} &= \frac{\partial L}{\partial \ddot{x}}, \\ p_x &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} = \frac{\partial L}{\partial \dot{x}} - \dot{p}_{\dot{x}}. \end{aligned} \quad (3)$$

With this notation, the variation equation is

$$\delta A(\sigma) = \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x}(t) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(t) + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}}(t) \right\} \delta x dt + p_x \delta x|_{t_0}^{t_1} + p_{\dot{x}} \delta \dot{x}|_{t_0}^{t_1}. \quad (4)$$

With an eye towards the Cartan form, it is convenient to reinterpret a variation as the Lie derivative with respect to a vector field. Let a variation $\sigma \mapsto \sigma + \delta\sigma : t \mapsto x(t) + \delta x(t)$ of σ be given by a vector field X on J^2 that is tangent to the fibres of the source map $J^2 \rightarrow [t_0, t_1] : (t, x, \dot{x}, \ddot{x}) \mapsto t$. In other words, if $X = X_x \frac{\partial}{\partial x}$, then

$$\delta x(t) = X_x(\sigma(t)).$$

Then the variation $j^2\sigma \mapsto j^2\sigma + \delta j^2\sigma$ is given by the prolongation

$$X^2 = X_x \frac{\partial}{\partial x} + X_{\dot{x}} \frac{\partial}{\partial \dot{x}} + X_{\ddot{x}} \frac{\partial}{\partial \ddot{x}}$$

of X to J^2 , where

$$X_{\dot{x}} = \frac{d}{dt} X_x \text{ and } X_{\ddot{x}} = \frac{d}{dt} X_{\dot{x}} = \frac{d^2}{dt^2} X_x.$$

In other words,

$$\delta \dot{x}(t) = X_{\dot{x}}(j^2 \sigma(t)) \text{ and } \delta \ddot{x}(t) = X_{\ddot{x}}(j^2(\sigma)).$$

With this identification,

$$\begin{aligned} \delta A(\sigma) &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x}(t) \delta x + \frac{\partial L}{\partial \dot{x}}(t) \delta \dot{x} + \frac{\partial L}{\partial \ddot{x}}(t) \delta \ddot{x} \right\} dt \\ &= \int_{t_0}^{t_1} \mathbb{L}_{X^2}(L dt) = \int_{t_0}^{t_1} X^2 \lrcorner d(L dt) \end{aligned} \quad (5)$$

because

$$\mathbb{L}_{X^2}(L dt) = X^2 \lrcorner d(L dt) + d(X^2 \lrcorner L dt),$$

and the assumption that X is tangent to the fibres of the source map implies that $X^2 \lrcorner L dt = 0$. Comparing equations (4) and (5) yields

$$\begin{aligned} \int_{t_0}^{t_1} X^2 \lrcorner d(L dt) &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} \right\} X_x dt + p_x X_x|_{t_0}^{t_1} + p_{\dot{x}} X_{\dot{x}}|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} \right\} X_x dt + \left\langle p_x dx + p_{\dot{x}} d\dot{x}, X^1 \right\rangle \Big|_{t_0}^{t_1}, \end{aligned} \quad (6)$$

where $\langle p_x dx + p_{\dot{x}} d\dot{x}, X^1 \rangle$ is the evaluation of a 1-form $p_x dx + p_{\dot{x}} d\dot{x}$ on the first jet bundle on the first jet prolongation X^1 of X (see proposition (2.6)). In equation (6) $p_x dx$ and $p_{\dot{x}} d\dot{x}$ are interpreted as one-forms on J^1 .

2.2. The Cartan form. The contact forms of the second jet bundle J^2 are

$$\theta_1 = dx - \dot{x} dt \text{ and } \theta_2 = d\dot{x} - \ddot{x} dt.$$

Their importance stems from the following

PROPOSITION 2.2. *A section $\sigma : [t_0, t_1] \rightarrow J^2 : t \mapsto (t, x(t), \dot{x}(t), \ddot{x}(t))$ is the jet extension of the section of its projection $[t_0, t_1] \rightarrow \mathbf{R}^n : t \mapsto (t, x(t))$ by the source map $J^2 \rightarrow [t_0, t_1] : (t, x, \dot{x}, \ddot{x}) \mapsto (t, x)$ if and only if $\sigma^* \theta_1 = 0$ and $\sigma^* \theta_2 = 0$.*

DEFINITION 2.3. The Cartan form corresponding to a Lagrangian L is the one-form Θ on J^3 given by

$$\Theta = L dt + p_x(dx - \dot{x} dt) + p_{\dot{x}}(d\dot{x} - \ddot{x} dt), \quad (7)$$

where p_x and $p_{\dot{x}}$ are the Ostrogradski momenta (3).

Observe that Θ may be written in the form

$$\Theta = p_x dx + p_{\dot{x}} d\dot{x} - H dt, \quad (8)$$

where

$$H = p_x \dot{x} + p_{\dot{x}} \ddot{x} - L \quad (9)$$

is the Hamiltonian of the theory. Since Θ differs from the Lagrange form $L dt$ by terms that are proportional to the contact forms, it follows that for any section σ the action $A(\sigma)$ can be expressed as the integral of Θ over $j^3\sigma$. In other words,

$$A(\sigma) = \int_{t_0}^{t_1} (L \circ j^2\sigma) dt = \int_{t_0}^{t_1} (j^2\sigma)^* L dt = \int_{t_0}^{t_1} (j^3\sigma)^* \Theta. \quad (10)$$

Therefore, the Cartan form Θ may be used instead of the Lagrange form $L dt$ to describe the variational problem under consideration. Other aspects of the Cartan form are discussed in [10].

DEFINITION 2.4. A Lagrangian L is *regular* if the matrix

$$\frac{\partial^2 L}{\partial \ddot{x}_j \partial \ddot{x}_i}$$

is non-singular.

THEOREM 2.5. Let γ be a section of the source map $J^3 \rightarrow [t_0, t_1]$ projecting to a section σ of $[t_0, t_1] \times \mathbf{R}^n \rightarrow [t_0, t_1]$ and let $j^3\sigma$ be the third jet extension of σ .

- (1) If $\gamma = j^3\sigma$, then σ satisfies the Euler-Lagrange equations if and only if the tangent bundle of the range of γ is contained in the kernel of $d\Theta$.
- (2) If the Lagrangian L is regular and the tangent bundle of the range of γ is contained in the kernel of $d\Theta$, then $\gamma = j^3\sigma$ and σ satisfies the Euler-Lagrange equations.

Proof. The exterior differential of Θ can be written as

$$d\Theta = \dot{p}_{\dot{x}} d\dot{x} \wedge dt + \frac{\partial L}{\partial x} dx \wedge dt + dp_x \wedge (dx - \dot{x} dt) + dp_{\dot{x}} \wedge (d\dot{x} - \ddot{x} dt),$$

because

$$\begin{aligned} d\Theta &= dL \wedge dt - p_x d\dot{x} \wedge dt - p_{\dot{x}} d\ddot{x} \wedge dt + dp_x \wedge (dx - \dot{x} dt) + dp_{\dot{x}} \wedge (d\dot{x} - \ddot{x} dt) \\ &= (L_{\ddot{x}} - p_{\dot{x}}) d\ddot{x} \wedge dt + (L_{\dot{x}} - p_x) d\dot{x} \wedge dt + L_x dx \wedge dt + \\ &\quad + dp_x \wedge (dx - \dot{x} dt) + dp_{\dot{x}} \wedge (d\dot{x} - \ddot{x} dt) \\ &= \dot{p}_{\dot{x}} d\dot{x} \wedge dt + L_x dx \wedge dt + dp_x \wedge (dx - \dot{x} dt) + dp_{\dot{x}} \wedge (d\dot{x} - \ddot{x} dt). \end{aligned}$$

A section $\gamma : t \mapsto \gamma(t) = (t, x(t), \dot{x}(t), \ddot{x}(t))$ of the source map projects to a section $\sigma : t \mapsto \sigma(t) = (t, x(t))$. Moreover,

$$T\gamma(\partial_t) = \frac{\partial}{\partial t} + \frac{dx(t)}{dt} \frac{\partial}{\partial x} + \frac{d\dot{x}(t)}{dt} \frac{\partial}{\partial \dot{x}} + \frac{d\ddot{x}(t)}{dt} \frac{\partial}{\partial \ddot{x}}$$

spans the tangent bundle of the range of γ , and

$$T\gamma(\partial_t)(p_x) = \dot{p}_x \text{ and } T\gamma(\partial_t)(p_{\dot{x}}) = \dot{p}_{\dot{x}}.$$

Hence,

$$\begin{aligned}
& T\gamma(\partial_t) \lrcorner d\Theta = \\
&= \dot{p}_{\dot{x}} \frac{d\dot{x}(t)}{dt} dt - \dot{p}_{\ddot{x}} d\dot{x} + \frac{\partial L}{\partial x} \frac{dx(t)}{dt} dt - \frac{\partial L}{\partial x} dx + p_{\dot{x}}(dx - \dot{x} dt) + \dot{p}_{\ddot{x}}(d\dot{x} - \ddot{x} dt) + \\
&\quad - \langle dx - \dot{x} dt, T\gamma(\partial_t) \rangle dp_x - \langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle dp_{\ddot{x}} \\
&= \dot{p}_{\dot{x}} \left(\frac{d\dot{x}(t)}{dt} - \ddot{x} \right) dt + \frac{\partial L}{\partial x} \left(\frac{dx(t)}{dt} - \dot{x} \right) dt + \left(\dot{p}_x - \frac{\partial L}{\partial x} \right) (dx - \dot{x} dt) + \\
&\quad - \langle dx - \dot{x} dt, T\gamma(\partial_t) \rangle dp_x - \langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle dp_{\ddot{x}}.
\end{aligned}$$

To prove the first statement observe that if $\gamma = j^3\sigma$, then

$$\begin{aligned}
\frac{d\dot{x}(t)}{dt} - \ddot{x} &= 0, \\
\frac{dx(t)}{dt} - \dot{x} dt &= 0, \\
\langle dx - \dot{x} dt, T\gamma(\partial_t) \rangle &= 0, \\
\langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle &= 0,
\end{aligned}$$

and

$$T\gamma(\partial_t) \lrcorner d\Theta = \left(\dot{p}_x - \frac{\partial L}{\partial x} \right) (dx - \dot{x} dt).$$

Hence, $T\gamma(\partial_t)$ is in the kernel of $d\Theta$ if and only if $\dot{p}_x - \frac{\partial L}{\partial x} = 0$. By definition of the Ostrogradski momenta,

$$\dot{p}_x - \frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) - \frac{\partial L}{\partial x},$$

and so $T\gamma(\partial_t)$ is in the kernel of $d\Theta$ if and only if the section σ satisfies the Euler-Lagrange equations.

To prove the second part, suppose that $T\gamma(\partial_t)$ is in the kernel of $d\Theta$. Then

$$\begin{aligned}
& \dot{p}_{\dot{x}} \left(\frac{d\dot{x}(t)}{dt} - \ddot{x} \right) dt + \frac{\partial L}{\partial x} \left(\frac{dx(t)}{dt} - \dot{x} \right) dt + \left(\dot{p}_x - \frac{\partial L}{\partial x} \right) (dx - \dot{x} dt) + \quad (11) \\
& - \langle dx - \dot{x} dt, T\gamma(\partial_t) \rangle dp_x - \langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle dp_{\ddot{x}} = 0.
\end{aligned}$$

Evaluating the left hand side on a vector $v \frac{\partial}{\partial \ddot{x}}$, where v is an arbitrary vector in \mathbf{R}^n , yields

$$\langle dx - \dot{x} dt, T\gamma(\partial_t) \rangle v \frac{\partial p_x}{\partial \ddot{x}} = 0,$$

because only p_x depends on \ddot{x} . Since

$$p_x = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} = \frac{\partial L}{\partial \dot{x}} - \frac{\partial^2 L}{\partial t \partial \ddot{x}} - \dot{x} \frac{\partial^2 L}{\partial x \partial \ddot{x}} - \ddot{x} \frac{\partial^2 L}{\partial \dot{x} \partial \ddot{x}} - \ddot{\ddot{x}} \frac{\partial^2 L}{\partial \ddot{x} \partial \ddot{x}},$$

it follows that

$$v \frac{\partial p_x}{\partial \ddot{x}} = -v \frac{\partial^2 L}{\partial \ddot{x} \partial \ddot{x}}.$$

Therefore,

$$\langle dx - \dot{x} dt, T\sigma(\partial_t) \rangle v \frac{\partial^2 L}{\partial \ddot{x} \partial \ddot{x}} = 0$$

for an arbitrary vector v . By assumption of regularity of the Lagrangian, the matrix $\left(\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}}\right)$ is non-singular. Hence,

$$\langle dx - \dot{x} dt, T\sigma(\partial_t) \rangle = 0,$$

which implies that $\frac{d\dot{x}(t)}{dt} - \ddot{x}(t) = 0$. Substituting these results into equation (11) yields

$$\dot{p}_{\dot{x}} \left(\frac{d\dot{x}(t)}{dt} - \ddot{x} \right) dt + \left(\dot{p} - \frac{\partial L}{\partial x} \right) (dx - \dot{x} dt) - \langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle dp_{\dot{x}} = 0. \quad (12)$$

Evaluating the left hand side of this equation on a vector $v \frac{\partial}{\partial \dot{x}}$, where v is an arbitrary vector in \mathbf{R}^n , yields

$$\langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle v \frac{\partial p_{\dot{x}}}{\partial \dot{x}} = 0.$$

Since

$$\frac{\partial p_{\dot{x}}}{\partial \dot{x}} = \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}},$$

$$\langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle v \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} = 0$$

for every vector v in \mathbf{R}^n . The assumed regularity of L implies that

$$\langle d\dot{x} - \ddot{x} dt, T\gamma(\partial_t) \rangle = 0,$$

so that $\frac{d\dot{x}(t)}{dt} - \ddot{x} = 0$. Hence, $\gamma = j^3 \sigma$ and equation (12) reads

$$\left(\dot{p}_x - \frac{\partial L}{\partial x} \right) (dx - \dot{x} dt) = 0,$$

which implies that σ satisfies the Euler-Lagrange equations. q.e.d.

2.3. Symmetries and conservation laws.

2.3.1. Symmetries of the Lagrange form. Consider an infinitesimal transformation in $(t_0, t_1) \times \mathbf{R}^n$ given by

$$\bar{t} = t + \epsilon \tau(t, x), \quad \bar{x}^i = x^i + \epsilon \xi^i(t, x).$$

It corresponds to a local one-parameter group of local diffeomorphisms generated by the vector field

$$X = \tau \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i}. \quad (13)$$

PROPOSITION 2.6. *The prolongations X^1 , X^2 and X^3 of the vector field X in equation (13) to the jet bundles J^1 , J^2 and J^3 , respectively, are*

$$X^1 = \tau \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}},$$

$$X^2 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{x}},$$

$$X^3 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{x}} + (\dddot{\xi} - 3\ddot{x}\dot{\tau} - 3\dot{x}\ddot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \dddot{x}}.$$

It remains to relate the prolongations of X to the contact forms $\theta_1 = dx - \dot{x} dt$ and $\theta_2 = d\dot{x} - \ddot{x} dt$.

PROPOSITION 2.7. *Let $I = [t_0, t_1]$. For a section σ of $I \times \mathbf{R}^n \rightarrow I$,*

$$j^1 \sigma^* \mathbb{L}_{X^1} \theta_1 = 0 \text{ and } j^2 \sigma^* \mathbb{L}_{X^2} \theta_2 = 0.$$

Proof. Since

$$X^1 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x} \dot{\tau}) \frac{\partial}{\partial \dot{x}},$$

then

$$\begin{aligned} \mathbb{L}_{X^1} \theta_1 &= X^1 \lrcorner d\theta_1 + d\langle \theta_1, X^1 \rangle \\ &= \left(\frac{\partial \xi}{\partial x} - \dot{x} \frac{\partial \tau}{\partial x} \right) (dx - \dot{x} dt). \end{aligned}$$

Therefore

$$\begin{aligned} j^1 \sigma^* \mathbb{L}_{X^1} \theta_1 &= j^1 \sigma^* (-\dot{\xi} dt + \dot{x} \dot{\tau} dt + d\xi - \dot{x} d\tau), \\ &= -\dot{\xi} dt + \dot{x} \dot{\tau} dt + \dot{\xi} dt - \dot{x} \dot{\tau} dt, \\ &= 0. \end{aligned}$$

On the other hand, since $\theta_2 = d\dot{x} - \ddot{x} dt$, and

$$X^2 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x} \dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\ddot{\xi} - 2\dot{x} \ddot{\tau} - \dot{x} \ddot{\tau}) \frac{\partial}{\partial \ddot{x}}$$

it follows that

$$\begin{aligned} \mathbb{L}_{X^2} \theta_2 &= X^2 \lrcorner d\theta_2 + d\langle \theta_2, X^2 \rangle \\ &= (d\dot{\xi} - \ddot{\xi} dt) - \dot{\tau}(d\dot{x} - \ddot{x} dt) - \ddot{x} \left\langle \frac{\partial \tau}{\partial x}, (dx - \dot{x} dt) \right\rangle \\ &\quad + \dot{x} \left\langle \frac{\partial}{\partial x} \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x} \dot{x} \right), (dx - \dot{x} dt) \right\rangle + \dot{x} \left\langle \frac{\partial \tau}{\partial x}, (d\dot{x} - \ddot{x} dt) \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} j^2 \sigma^* \mathbb{L}_{X^2} \theta_2 &= j^2 \sigma^* \left\{ (d\dot{\xi} - \ddot{\xi} dt) - \dot{\tau}(d\dot{x} - \ddot{x} dt) - \ddot{x} \left\langle \frac{\partial \tau}{\partial x}, (dx - \dot{x} dt) \right\rangle \right\} + \\ &\quad + j^2 \sigma^* \left\{ \dot{x} \left\langle \frac{\partial}{\partial x} \left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x} \dot{x} \right), (dx - \dot{x} dt) \right\rangle + \dot{x} \left\langle \frac{\partial \tau}{\partial x}, (d\dot{x} - \ddot{x} dt) \right\rangle \right\} \\ &= 0. \end{aligned}$$

q.e.d.

Let σ be section of $I \times \mathbf{R}^n \rightarrow I$, where $I = [t_0, t_1]$ and

$$A(\sigma) = \int_I (L \circ j^2 \sigma) dt = \int_I j^2 \sigma^* (L dt) = \int_{j^2 \sigma(I)} L dt. \quad (14)$$

PROPOSITION 2.8. *The action integral (14) is invariant under the one-parameter local group $\exp tX^2$ of local diffeomorphisms of J^2 generated by X^2 if*

$$j^2\sigma^*(\mathbb{L}_{X^2}(L dt)) = 0.$$

Moreover, $\frac{d}{dt}A(\exp tX(\sigma))|_{t=0}$ for every section σ , if and only if $\mathbb{L}_{X^2}(L dt) = 0$.

Proof. Consider a vector field X on $I \times \mathbf{R}^n$. Denote by $\exp tX$ the local one-parameter group of local diffeomorphisms of $I \times \mathbf{R}^n$ generated by X , and by $\exp tX^2$ the local one-parameter group of local diffeomorphisms of J^2 generated by the prolongation X^2 of X to J^2 . Then,

$$A(\exp tX(\sigma)) = \int_{\exp tX^2(j^2\sigma(I))} L dt.$$

The change of variables theorem asserts that $\int_{\phi_t(c)} \omega = \int_c \phi_t^* \omega$ for any form ω , chain c and a one-parameter group ϕ_t diffeomorphisms. Hence,

$$\int_{\exp tX^2(j^2\sigma(I))} L dt = \int_{j^2\sigma(I)} (\exp tX^2)^* L dt.$$

Therefore,

$$A(\exp tX(\sigma)) = \int_{j^2\sigma(I)} (\exp tX^2)^* L dt.$$

Now, differentiating under the integral sign with respect to t ,

$$\frac{d}{dt}A(\exp tX(\sigma)) = \int_{j^2\sigma(I)} \frac{d}{dt}(\exp tX^2)^* L dt = \int_{j^2\sigma(I)} (\exp tX^2)^* \mathbb{L}_{X^2} L dt,$$

and setting $t = 0$,

$$\frac{d}{dt}A(\exp tX(\sigma))|_{t=0} = \int_{j^2\sigma(I)} \mathbb{L}_{X^2} L dt = \int_I j^2\sigma^*(\mathbb{L}_{X^2}(L dt)),$$

it follows that if $j^2\sigma^*\mathbb{L}_{X^2} L dt = 0$, then $\frac{d}{dt}A(\exp tX(\sigma))|_{t=0} = 0$. Moreover, $\frac{d}{dt}A(\exp tX(\sigma))|_{t=0} = 0$ for every section σ , if and only if $\mathbb{L}_{X^2} L dt = 0$. q.e.d.

DEFINITION 2.9. A vector field X on $I \times \mathbf{R}^n$ is an infinitesimal symmetry of the Lagrange form $L dt$ if $\mathbb{L}_{X^2}(L dt) = 0$.

LEMMA 2.10. *The Lie derivative of the Lagrange form $L dt$ with respect to X^2 is*

$$\mathbb{L}_{X^2}(L dt) = \left(\tau \frac{\partial L}{\partial t} + \xi^i \frac{\partial L}{\partial x^i} + (\xi - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial L}{\partial \ddot{x}} \right) dt + L d\tau.$$

Hence, for every section σ of $I \times \mathbf{R}^n \rightarrow I$,

$$j^2\sigma^*(\mathbb{L}_{X^2}(L dt)) = \left(\tau \frac{\partial L}{\partial t} + \xi^i \frac{\partial L}{\partial x^i} + (\xi - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial L}{\partial \ddot{x}} + L\dot{\tau} \right) dt.$$

Proof. The Lie derivative of the Lagrangian form $L dt$ with respect to X^2 is

$$\begin{aligned}\mathfrak{L}_{X^2}(L dt) &= X^2 \lrcorner dL \wedge dt + d(X^2 \lrcorner L dt) \\ &= X^2 \lrcorner \left(\left(\frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x^i} dx + \frac{\partial L}{\partial \dot{x}} d\dot{x} + \frac{\partial L}{\partial \ddot{x}} d\ddot{x} \right) \wedge dt \right) + d(X^2 \lrcorner L dt) \\ &= X^2 \lrcorner \left(\left(\frac{\partial L}{\partial x^i} dx + \frac{\partial L}{\partial \dot{x}} d\dot{x} + \frac{\partial L}{\partial \ddot{x}} d\ddot{x} \right) \wedge dt \right) + d(X^2 \lrcorner L dt)\end{aligned}$$

because $dt \wedge dt = 0$. Hence,

$$\begin{aligned}\mathfrak{L}_{X^2}(L dt) &= \left(\xi^i \frac{\partial L}{\partial x^i} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial L}{\partial \ddot{x}} \right) dt + \\ &\quad - \tau \left(\frac{\partial L}{\partial x^i} dx + \frac{\partial L}{\partial \dot{x}} d\dot{x} + \frac{\partial L}{\partial \ddot{x}} d\ddot{x} \right) + L d\tau + \tau dL \\ &= \left(\tau \frac{\partial L}{\partial t} + \xi^i \frac{\partial L}{\partial x^i} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial L}{\partial \ddot{x}} \right) dt + L d\tau.\end{aligned}$$

q.e.d.

LEMMA 2.11. (Noether identities.) The equation $j^2 \sigma^*(\mathfrak{L}_{X^2}(L dt)) = 0$ is equivalent to

$$\begin{aligned}&\frac{d}{dt} \left(L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}) \right) = \\ &= - \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}).\end{aligned}\tag{15}$$

Proof. The details of this routine but lengthy calculation may be found in [16].

q.e.d.

REMARK 2.12. The Noether identity is essentially the extension of the equation $j^2 \sigma^*(\mathfrak{L}_{X^2}(L dt)) = 0$ to the fourth jet bundle. More precisely, if $\pi_{4,2} : J^4 \rightarrow J^2$ is the natural projection and X^4 is the prolongation of X to J^4 , then

$$\begin{aligned}j^4 \sigma^*(\pi_{4,2}^*(\mathfrak{L}_{X^2}(L dt))) &= j^4 \sigma^* \left(\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) \right) + \\ &\quad + j^4 \sigma^* \left(\frac{d}{dt} \left(L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}) \right) \right).\end{aligned}\tag{16}$$

An immediate corollary of the Noether identity is the following conservation law.

THEOREM 2.13. (First Noether theorem.) To every infinitesimal symmetry $X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x}$ of the Lagrange form $L dt$, there corresponds a conserved quantity

$$\mathcal{J}_X = L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}).\tag{17}$$

That is, \mathcal{J}_X is constant along solutions of the Euler-Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) = 0.$$

In other words, if σ satisfies the Euler-Lagrange equations, then $j^3\sigma^*(X^3 \lrcorner \Theta)$ is constant.

EXAMPLE 2.14. (Conservation of linear momentum.) If the Lagrangian L does not depend on the coordinate x^i , then $X = \frac{\partial}{\partial x^i}$ is an infinitesimal symmetry and the momentum

$$\mathcal{J}_{\frac{\partial}{\partial x^i}} = \frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right)$$

is conserved.

EXAMPLE 2.15. (Conservation of energy) If the Lagrangian L does not depend on the parameter t , then $X = \frac{\partial}{\partial t}$ is an infinitesimal symmetry and the energy

$$H = p_x \dot{x} + p_{\ddot{x}} \ddot{x} - L = - \mathcal{J}_{\frac{\partial}{\partial t}}$$

is conserved.

Proof.

$$\begin{aligned} \mathcal{L}_{\frac{\partial}{\partial t}}(L dt) &= \frac{\partial}{\partial t} \lrcorner d(L dt) + d\left(\frac{\partial}{\partial t} \lrcorner L dt\right) \\ &= \frac{\partial}{\partial t} \lrcorner dL \wedge dt + dL \\ &= -\left(\frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial \dot{x}} d\dot{x} + \frac{\partial L}{\partial \ddot{x}} d\ddot{x}\right) + \left(\frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial \dot{x}} d\dot{x} + \frac{\partial L}{\partial \ddot{x}} d\ddot{x}\right) \\ &= \frac{\partial L}{\partial t} dt. \end{aligned}$$

Hence, $\frac{\partial L}{\partial t} = 0$ implies that $\frac{\partial}{\partial t}$ is an infinitesimal symmetry of the Lagrange form $L dt$. q.e.d.

Other conserved quantities will be discussed later.

REMARK 2.16. There is a vast amount of work on symmetry principles and conservation laws following Noether's fundamental paper [17]. Three works in particular are noteworthy: the monographs by Logan [16] and Kosmann-Schwarzbach [9], and a review by Krupkova [11].

2.3.2. *The Cartan form approach.* Recall that the Cartan form Θ is

$$\begin{aligned} \Theta &= L dt + p_x \theta_1 + p_{\ddot{x}} \theta_2 \\ &= L dt + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) (dx - \dot{x} dt) + \frac{\partial L}{\partial \ddot{x}} (d\dot{x} - \ddot{x} dt). \end{aligned} \tag{18}$$

LEMMA 2.17. For every section $\sigma : I \rightarrow I \times \mathbf{R}^n$, and each vector field $X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x}$ on $I \times \mathbf{R}^n$,

$$j^3\sigma^*(\mathcal{L}_{X^3}\Theta) = j^2\sigma^*(\mathcal{L}_{X^2}(L dt)).$$

In particular, if

$$j^2\sigma^*(\mathcal{L}_{X^2}(L dt)) = 0,$$

then,

$$j^3\sigma^*(\mathcal{L}_{X^3}\Theta) = 0.$$

Proof. By proposition 2.7,

$$j^2 \sigma^* \mathfrak{L}_{X^2} \theta_2 = 0 \quad \text{and} \quad j^1 \sigma^* \mathfrak{L}_{X^1} \theta_1 = 0.$$

Lifting these equations to J^3 , we get

$$j^3 \sigma^* \mathfrak{L}_{X^3} \pi_{3,2}^* \theta_2 = 0 \quad \text{and} \quad j^3 \sigma^* \mathfrak{L}_{X^3} \pi_{3,1}^* \theta_1 = 0.$$

Equation (18) written in terms of pull-backs of the jet projections $\pi_{j,i}$ reads

$$\Theta = \pi_{3,2}^*(L dt) + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) \pi_{3,1}^* \theta_1 + \frac{\partial L}{\partial \ddot{x}} \pi_{3,2}^* \theta_2,$$

where we consider the coefficients $\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right)$ and $\frac{\partial L}{\partial \ddot{x}}$ as functions on J^3 . Hence,

$$\begin{aligned} j^3 \sigma^*(\mathfrak{L}_{X^3} \Theta) &= j^3 \sigma^*(\mathfrak{L}_{X^3}(\pi_{3,2}^*(L dt))) + j^3 \sigma^* \left(\mathfrak{L}_{X^3} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) \right) j^3 \sigma^* \pi_{3,1}^* \theta_1 \\ &\quad + j^3 \sigma^* \left(\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) \right) j^3 \sigma^*(\mathfrak{L}_{X^3} \pi_{3,1}^* \theta_1) + \\ &\quad + j^3 \sigma^* \left(\mathfrak{L}_{X^3} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) j^3 \sigma^*(\pi_{3,2}^* \theta_2) + j^3 \sigma^* \left(\mathfrak{L}_{X^3} \frac{\partial L}{\partial \ddot{x}} \right) j^3 \sigma^*(\mathfrak{L}_{X^3} \pi_{3,2}^* \theta_2). \end{aligned}$$

But,

$$\begin{aligned} j^3 \sigma^*(\mathfrak{L}_{X^3}(\pi_{3,2}^*(L dt))) &= j^2 \sigma^*(\mathfrak{L}_{X^2}(L dt)), \\ j^3 \sigma^* \pi_{3,1}^* \theta_1 &= j^1 \sigma^* \theta_1 = 0, \\ j^3 \sigma^*(\pi_{3,2}^* \theta_2) &= j^2 \sigma^* \theta_2 = 0, \\ j^3 \sigma^*(\mathfrak{L}_{X^3} \pi_{3,1}^* \theta_1) &= j^3 \sigma^*(\pi_{3,1}^*(\mathfrak{L}_{X^1} \theta_1)) = j^1 \sigma^*(\mathfrak{L}_{X^1} \theta_1) = 0, \\ j^3 \sigma^*(\mathfrak{L}_{X^3} \pi_{3,2}^* \theta_2) &= j^3 \sigma^*(\pi_{3,2}^*(\mathfrak{L}_{X^2} \theta_2)) = j^2 \sigma^*(\mathfrak{L}_{X^2} \theta_2) = 0. \end{aligned}$$

Hence,

$$j^3 \sigma^*(\mathfrak{L}_{X^3} \Theta) = j^2 \sigma^*(\mathfrak{L}_{X^2}(L dt)).$$

q.e.d.

PROPOSITION 2.18. *If $X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x}$ is an infinitesimal symmetry of the Lagrange form $L dt$ then, for every section σ of $I \times \mathbf{R}^n \rightarrow I$,*

$$j^3 \sigma^* \mathcal{J}_X = j^3 \sigma^*(X^3 \lrcorner \Theta), \quad (19)$$

where X^3 is the prolongation of X to J^3 . If σ satisfies the Euler-Lagrange equations for L , then $j^3 \sigma^*(X^3 \lrcorner \Theta)$ is constant.

Proof. Omitting pull-backs by $j^3\sigma$ for the sake of transparency, we may write

$$\begin{aligned}
(X^3 \lrcorner \Theta) &= \\
&= \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{x}} + (\ddot{\xi} - 3\ddot{x}\dot{\tau} - 3\ddot{x}\ddot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{\ddot{x}}} \\
&\quad \lrcorner \left(L dt + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) (dx - \dot{x} dt) + \frac{\partial L}{\partial \ddot{x}} (d\dot{x} - \ddot{x} dt) \right) \\
&= \tau \left(L - \dot{x} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) - \ddot{x} \frac{\partial L}{\partial \ddot{x}} \right) + \xi \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \ddot{x}} \\
&= L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau\dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau\dot{x}) = \mathcal{J}_X.
\end{aligned} \tag{20}$$

Hence, $j^3\sigma^*(X^3 \lrcorner \Theta) = j^3\sigma^*\mathcal{J}_X$ and is a constant if σ satisfies the Euler-Lagrange equations. Suppose that $\gamma : I \rightarrow J^3$ is a section of the source map such that its tangent $T\gamma(I)$ is contained in $\ker d\Theta$. Since $\mathcal{L}_Z\Theta = Z \lrcorner d\Theta + d(Z \lrcorner \Theta)$, it follows that for any infinitesimal symmetry Z of Θ , we have

$$d\gamma^*(Z \lrcorner \Theta) = -\gamma^*d(Z \lrcorner \Theta) = -\gamma^*\mathcal{L}_Z\Theta = 0,$$

which implies that $\gamma^*(Z \lrcorner \Theta)$ is constant. This holds for any section γ with $T\gamma(I) \subset \ker \Theta$. Moreover, if $\gamma = j^3\sigma$, where σ satisfies the Euler-Lagrange equations, and $Z = X^3$, where X is an infinitesimal symmetry of the Lagrange form $L dt$, then lemma (2.17) implies that

$$\gamma^*\mathcal{L}_Z\Theta = j^2\sigma^*\mathcal{L}_X L dt. \tag{21}$$

Hence, if $j^2\sigma^*(\mathcal{L}_{X^2}(L dt)) = 0$, then $j^3\sigma^*(\mathcal{L}_{X^3}\Theta) = 0$, and $j^3\sigma^*(X^3 \lrcorner \Theta)$ is constant. Moreover, if $X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x}$, then

$$\begin{aligned}
(X^3 \lrcorner \Theta) &= \\
&= \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial}{\partial \dot{x}} + (\ddot{\xi} - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{x}} + (\ddot{\xi} - 3\ddot{x}\dot{\tau} - 3\ddot{x}\ddot{\tau} - \dot{x}\ddot{\tau}) \frac{\partial}{\partial \ddot{\ddot{x}}} \\
&\quad \lrcorner \left(L dt + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) (dx - \dot{x} dt) + \frac{\partial L}{\partial \ddot{x}} (d\dot{x} - \ddot{x} dt) \right) \\
&= \tau \left(L - \dot{x} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) - \ddot{x} \frac{\partial L}{\partial \ddot{x}} \right) + \xi \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) + (\dot{\xi} - \dot{x}\dot{\tau}) \frac{\partial L}{\partial \ddot{x}} \\
&= L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau\dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau\dot{x})
\end{aligned} \tag{22}$$

q.e.d.

Equation (19) gives a simple way of finding constants of motion corresponding to symmetries of the Lagrange form.

EXAMPLE 2.19. If $x = (x^i)$ are Cartesian coordinates in \mathbf{R}^n , then the action of $\text{SO}(n)$ on J^3 is generated by vector fields

$$X_{ij}^3 = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} + \dot{x}^i \frac{\partial}{\partial \dot{x}^j} - \dot{x}^j \frac{\partial}{\partial \dot{x}^i} + \ddot{x}^i \frac{\partial}{\partial \ddot{x}^j} - \ddot{x}^j \frac{\partial}{\partial \ddot{x}^i} + \ddot{\ddot{x}}^i \frac{\partial}{\partial \ddot{\ddot{x}}^j} - \ddot{\ddot{x}}^j \frac{\partial}{\partial \ddot{\ddot{x}}^i}.$$

Hence, for a section σ of $I \times \mathbf{R}^n \rightarrow I$,

$$\begin{aligned} j^3 \sigma^*(\mathcal{J}_{X_{ij}}) &= j^3 \sigma^*(X_{ij}^3 \lrcorner \Theta) = \\ &= j^3 \sigma^*(x^i p_{x^j} - x^j p_{x^i} + \dot{x}^i p_{\dot{x}^j} - \dot{x}^j p_{\dot{x}^i}). \end{aligned}$$

In the following we omit the symbol $j^3 \sigma^*$, and write

$$\mathcal{J}_{X_{ij}} = x^i p_{x^j} - x^j p_{x^i} + \dot{x}^i p_{\dot{x}^j} - \dot{x}^j p_{\dot{x}^i}.$$

If L is invariant under the action of $\text{SO}(3)$ on J^2 , then $\mathcal{J}_{X_{ij}}$ is constant on solutions of the Euler-Lagrange equations.

This suggests that there may be additional conserved quantities coming from symmetries of the Cartan form that are not symmetries of the Lagrange form.

DEFINITION 2.20. An infinitesimal symmetry of the Cartan form Θ is a vector field Z on J^3 such that $\mathfrak{L}_Z \Theta = 0$.

Set

$$\mathcal{J}_Z = Z \lrcorner \Theta$$

for each infinitesimal symmetry Z of the Cartan form Θ .

THEOREM 2.21. Let Z be an infinitesimal symmetry of the Cartan form and let $\gamma : I \rightarrow J^3$ be a section of the source map. If $T\gamma(I)$ is contained in $\ker d\Theta$, then

$$\gamma^* \mathcal{J}_Z = \gamma^*(Z \lrcorner \Theta)$$

is constant. In particular, if σ satisfies the Euler-Lagrange equations, then $j^3 \sigma^* \mathcal{J}_Z$ is a constant.

Proof. Since

$$\mathfrak{L}_Z \Theta = Z \lrcorner d\Theta + d(Z \lrcorner \Theta),$$

it follows that for any infinitesimal symmetry Z of Θ , and any section $\gamma : I \rightarrow J^3$ of the source map, that

$$d\gamma^*(Z \lrcorner \Theta) = -\gamma^*(Z \lrcorner d\Theta).$$

If $T\gamma(I)$ is contained in $\ker d\Theta$, then $\gamma^*(Z \lrcorner d\Theta) = 0$ and $\gamma^*(Z \lrcorner \Theta)$ is constant. q.e.d.

2.3.3. Symmetries up to a differential. The Cartan form may yield more conserved quantities than those that follow directly from the Lagrangian approach. However, even more conserved quantities may arise if the notion of symmetry is relaxed somewhat.

DEFINITION 2.22. A vector field X on $I \times \mathbf{R}$ is a *symmetry up to a differential* of the Lagrange form $L dt$ if there exists a function F on J^2 such that

$$\mathfrak{L}_{X^2}(L dt) = -dF,$$

where X^2 is the prolongation of X to J^2 .

PROPOSITION 2.23. *If a vector field $X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x}$ on $I \times \mathbf{R}^n$ satisfies the condition*

$$\mathbf{f}_{X^2}(L dt) = -dF, \quad (23)$$

where X^2 is the prolongation of X to J^2 , and F is a function on J^2 , then

$$\mathcal{J}_X = F + L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}) \quad (24)$$

is constant along solutions of the Euler-Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) = 0.$$

Proof. The pull-back of equation (23) by the jet bundle projection $\pi_{4,2} : J^4 \rightarrow J^2$ gives

$$\pi_{4,2}^* \mathbf{f}_{X^2}(L dt) = -\pi_{4,2}^* dF.$$

Therefore, for a section σ of $I \times \mathbf{R}^n$,

$$j^4 \sigma^* (\pi_{4,2}^* \mathbf{f}_{X^2}(L dt)) = -j^4 \sigma^* (\pi_{4,2}^* dF) = -j^2 \sigma^* dF.$$

Taking into account the form of the Noether identity given in equation (16) yields

$$\begin{aligned} -j^2 \sigma^* dF &= j^4 \sigma^* \left(\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \right. \\ &\quad \left. + j^4 \sigma^* \left(\frac{d}{dt} \left(L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}) \right) \right) \right). \end{aligned}$$

If σ satisfies the Euler-Lagrange equations, then

$$j^4 \sigma^* \left(\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) \right) = 0$$

and consequently

$$\mathcal{J}_X = F + \left(L\tau + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) (\xi - \tau \dot{x}) + \frac{\partial L}{\partial \ddot{x}} \frac{d}{dt} (\xi - \tau \dot{x}) \right)$$

is constant along σ .

q.e.d.

EXAMPLE 2.24. Probably the most well-known example of this sort of behaviour occurs in the first-order theory as the Runge-Lenz vector in the Kepler problem. In this case there is a symmetry of the dynamical system that is not lifted from the configuration space, and this implies that the Lagrangian is not invariant under the action of the symmetry group, but changes by a total derivative. This is discussed in [15].

In a similar way, infinitesimal symmetries up to a differential of the Cartan form are defined as

DEFINITION 2.25. A vector field Z on J^3 is a *symmetry up to a differential* of the Cartan form Θ if there exists a function F on J^3 such that

$$\mathbf{f}_Z \Theta = -dF. \quad (25)$$

PROPOSITION 2.26. *If $\mathbb{L}_Z\Theta = -dF$, then for a section $\gamma : I \rightarrow J^3$ such that $T(\gamma(I))$ is contained in $\ker d\Theta$, the function $F + \langle \Theta, Z \rangle$ is constant along γ . In particular, $F + \langle \Theta, Z \rangle$ is constant along the jet extensions of sections σ of $I \times \mathbf{R}^n$ that satisfy the Euler-Lagrange equations.*

Proof. Since $\mathbb{L}_Z\Theta = Z \lrcorner d\Theta + d\langle \Theta, Z \rangle$, equation (25) gives

$$Z \lrcorner d\Theta = -d(F + \langle \Theta, Z \rangle).$$

Hence,

$$\gamma^*(Z \lrcorner d\Theta) = -\gamma^*d(F + \langle \Theta, Z \rangle) = -d\gamma^*(F + \langle \Theta, Z \rangle).$$

However, $\gamma^*(Z \lrcorner d\Theta) = 0$ if $T(\gamma(I))$ is contained in $\ker d\Theta$. Therefore, $d\gamma^*(F + \langle \Theta, Z \rangle) = 0$, which implies that $F + \langle \Theta, Z \rangle$ is constant along γ .

If a section σ of $I \times \mathbf{R}^n$ satisfies the Euler-Lagrange equations, then $T(j^3\sigma(I))$ is in the kernel of $d\Theta$. Therefore, $F + \langle \Theta, Z \rangle$ is constant along $j^3\sigma$. q.e.d.

2.4. Parametrization invariance. Let $\text{Diff}_+ \mathbf{R}$ be the group of orientation preserving diffeomorphisms of the real line \mathbf{R} . Then, for every $\varphi \in \text{Diff}_+ \mathbf{R}$, $\dot{\varphi}(t) > 0$ for all t . Each $\varphi \in \text{Diff}_+ \mathbf{R}$ gives rise to another diffeomorphism

$$\varphi^0 : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R} \times \mathbf{R}^n : (t, x) \mapsto \varphi^0(t, x) = (\varphi(t), x).$$

The prolongations of φ^0 to jet bundles can be written as follows

$$\begin{aligned} \varphi^1 & : J^1 \rightarrow J^1 : (t, x, \dot{x}) \mapsto \varphi^1(t, x, \dot{x}) = \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)} \right), \\ \varphi^2 & : J^2 \rightarrow J^2 : (t, x, \dot{x}, \ddot{x}) \mapsto \varphi^2(t, x, \dot{x}, \ddot{x}) = \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)}, \frac{\ddot{x}}{\dot{\varphi}(t)^2} - \dot{x} \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^3} \right), \\ \varphi^3 & : J^3 \rightarrow J^3 : (t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \mapsto \varphi^3(t, x, \dot{x}, \ddot{x}) = \\ & = \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)}, \frac{\ddot{x}}{\dot{\varphi}(t)^2} - \dot{x} \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^3}, \frac{\ddot{\ddot{x}}}{\dot{\varphi}(t)^3} - 3\ddot{x} \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^4} - \dot{x} \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^4} + 3\dot{x} \frac{\ddot{\varphi}(t)^2}{\dot{\varphi}(t)^5} \right). \end{aligned} \tag{26}$$

PROPOSITION 2.27. *For $\varphi \in \text{Diff}_+ \mathbf{R}$,*

$$\begin{aligned} \varphi^{1*}\theta_1 & = \theta_1, \\ \varphi^{2*}\theta_2 & = \frac{1}{\dot{\varphi}}\theta_2. \end{aligned}$$

Proof. Let $I = (-\infty, \infty)$. The contact forms are $\theta_1 = dx - \dot{x} dt$ and $\theta_2 = d\dot{x} - \ddot{x} dt$. Since

$$\varphi^1(t, x, \dot{x}) = (\varphi(t), x, \dot{x}/\dot{\varphi}(t)),$$

it follows that

$$\varphi^{1*}\theta_1 = dx - x' dt = dx - \frac{\dot{x}}{\dot{\varphi}(t)} d(\varphi(t)) = dx - \frac{\dot{x}}{\dot{\varphi}(t)} \dot{\varphi}(t) dt = dx - \dot{x} dt = \theta_1.$$

Similarly,

$$\varphi^2 : J^2 \rightarrow J^2 : (t, x, \dot{x}, \ddot{x}) \mapsto \varphi^2(t, x, \dot{x}, \ddot{x}) = \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)}, \frac{\ddot{x}}{\dot{\varphi}(t)^2} - \dot{x} \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^3} \right)$$

implies

$$\begin{aligned}
\varphi^{2*}\theta_2 &= \varphi^{2*}(dx' - x'' dt) \\
&= d\left(\frac{\dot{x}}{\dot{\varphi}(t)}\right) - \left(\frac{\ddot{x}}{\dot{\varphi}(t)^2} - \dot{x}\frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^3}\right)d(\varphi(t)) \\
&= \frac{d\dot{x}}{\dot{\varphi}(t)} - \frac{\dot{x}}{\dot{\varphi}(t)^2}\ddot{\varphi}(t)dt - \left(\frac{\ddot{x}}{\dot{\varphi}(t)^2} - \dot{x}\frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^3}\right)\dot{\varphi}(t)dt \\
&= \frac{d\dot{x}}{\dot{\varphi}(t)} - \frac{\dot{x}}{\dot{\varphi}(t)^2}\ddot{\varphi}(t)dt - \frac{\ddot{x}}{\dot{\varphi}(t)}dt + \dot{x}\frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)^2}dt \\
&= \frac{1}{\dot{\varphi}(t)}(d\dot{x} - \ddot{x}dt) = \frac{1}{\dot{\varphi}(t)}\theta_2.
\end{aligned}$$

q.e.d.

A one-parameter subgroup $\varphi_\varepsilon : t \mapsto \bar{t} = \varphi_\varepsilon(t)$ of $\text{Diff}_+ \mathbf{R}$ is generated by a vector field $X_\tau = \tau(t)\partial_t$, where

$$\tau(t) = \left. \frac{\partial \varphi_\varepsilon(t)}{\partial \varepsilon} \right|_{\varepsilon=0}$$

is an arbitrary smooth function on R with $\dot{\tau}(t) \neq 0$ for all $t \in \mathbf{R}$. The Lie algebra of the group $\text{Diff}_+ \mathbf{R}$ is the collection of vector fields

$$\text{diff}_+ \mathbf{R} = \{X_\tau = \tau(t)\partial_t \mid \tau \in C^\infty(\mathbf{R}), \text{ and } \dot{\tau}(t) \neq 0 \text{ for all } t\}$$

with the Lie bracket

$$[\tau_1(t)\partial_t, \tau_2(t)\partial_t] = (\tau_1(t)\dot{\tau}_2(t) - \tau_2(t)\dot{\tau}_1(t))\partial_t.$$

DEFINITION 2.28. The variational problem with the Lagrangian L is *parametrization invariant* if the Lagrange form $L dt$ is invariant under the action of $\text{Diff}_+ \mathbf{R}$ on J^2 .

REMARK 2.29. Suppose that the Lagrange form $L dt$ is $\text{Diff}_+ \mathbf{R}$ -invariant. This implies that for $X_\tau = \tau(t)\partial_t$, the Lagrange form $L dt$ is invariant under the one-parameter subgroup of $\text{Diff}_+ \mathbf{R}$ generated by X_τ . By Theorem 2.18, $\mathcal{J}_{X_\tau} = \langle \Theta, X_\tau^3 \rangle$ is constant on solutions of the Euler-Lagrange equations.

THEOREM 2.30. *If the Lagrange form $L dt$ is $\text{Diff}_+ \mathbf{R}$ -invariant, then*

$$j^3\sigma^* \mathcal{J}_{X_\tau} = 0 \tag{27}$$

for all $X_\tau \in \text{diff}_+ \mathbf{R}$ and all solutions σ of the Euler-Lagrange equations.

Proof. Recall that $\Theta = p_x dx + p_{\dot{x}} d\dot{x} - H dt$. Omitting pull-backs by $j^3\sigma$ for the sake of cleanliness,

$$\begin{aligned}
\mathcal{J}_{X_\tau} = \langle \Theta, X_\tau^3 \rangle &= \left\langle p dx + p_{\dot{x}} d\dot{x} - H dt, \tau \frac{\partial}{\partial t} - \dot{x} \dot{\tau} \frac{\partial}{\partial \dot{x}} \right\rangle \\
&= -\langle p_{\dot{x}}, \dot{x} \rangle \dot{\tau} - H \tau.
\end{aligned} \tag{28}$$

If σ satisfies the Euler-Lagrange equations, then $j^3\sigma^* \mathcal{J}_{X_\tau}$ is constant.

Take two points, $t_0 < t_1$ in I , and consider two other vector fields X_{τ_1} and X_{τ_2} in $\text{diff}_+ \mathbf{R}$ such that

$$\begin{aligned}\tau(t_0) &= \tau_1(t_0) = \tau_2(t_0) \text{ and } \dot{\tau}(t_0) = \dot{\tau}_1(t_0) = \dot{\tau}_2(t_0), \\ \tau(t_1) &\neq \tau_1(t_1) = \tau_2(t_1) \text{ and } \dot{\tau}(t_1) = \dot{\tau}_1(t_1) \neq \dot{\tau}_2(t_1), \\ \tau(t_2) &\neq \tau_1(t_2) = \tau_2(t_2) \text{ and } \dot{\tau}(t_2) = \dot{\tau}_1(t_2) \neq \dot{\tau}_2(t_2).\end{aligned}$$

Then, $\mathcal{J}_{X_\tau}(t)$, $\mathcal{J}_{X_{\tau_1}}(t)$ and $\mathcal{J}_{X_{\tau_2}}(t)$ are constant along $j^3\sigma$. Moreover, the assumption that $\tau(t_0) = \tau_1(t_0) = \tau_2(t_0)$, $\dot{\tau}(t_0) = \dot{\tau}_1(t_0) = \dot{\tau}_2(t_0)$ and equation (28) imply that $\mathcal{J}_{X_\tau}(t) = \mathcal{J}_{X_{\tau_1}}(t) = \mathcal{J}_{X_{\tau_2}}(t)$ for all t . Therefore, $\mathcal{J}_{X_{\tau_1}}(t) - \mathcal{J}_{X_\tau}(t) = 0$ and $\mathcal{J}_{X_{\tau_2}}(t) - \mathcal{J}_{X_\tau}(t) = 0$ for all t . Using equation (28) and setting $t = t_1$ yields

$$\begin{aligned}p_x(t_1)\dot{x}(t_1)\dot{\tau}_1(t_1) + H(t_1)\tau_1(t_1) - p_x(t_1)\dot{x}(t_1)\dot{\tau}(t_1) - H(t_1)\tau(t_1) &= 0 \\ p_x(t_1)\dot{x}(t_1)\dot{\tau}_2(t_1) + H(t_1)\tau_2(t_1) - p_x(t_1)\dot{x}(t_1)\dot{\tau}(t_1) - H(t_1)\tau(t_1) &= 0.\end{aligned}$$

Since $\tau(t_1) \neq \tau_1(t_1)$ and $\dot{\tau}(t_1) = \dot{\tau}_1(t_1)$, the first equation above yields $H(t_1)(\tau(t_1) - \tau_1(t_1)) = 0$, which implies that $H(t_1) = 0$. Similarly, the assumption that $\tau_1(t_1) = \tau_2(t_1)$ and $\dot{\tau}_1(t_1) \neq \dot{\tau}_2(t_1)$ together with the second equation above yield $p_x(t_1)\dot{x}(t_1) = 0$. Since t_1 is an arbitrary point in I different from t_0 , it follows that

$$H(t) = 0 \text{ and } p_x(t)\dot{x}(t) = 0 \text{ for all } t \in I. \quad (29)$$

Substituting this result into equation 28 gives (27). q.e.d.

REMARK 2.31. Equations (29), rewritten in terms of the configuration variables read

$$\begin{aligned}\dot{x} \frac{\partial L}{\partial \ddot{x}} &= 0, \\ \dot{x} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} p_{\dot{x}} \right) + \ddot{x} \frac{\partial L}{\partial \ddot{x}} - L &= 0.\end{aligned} \quad (30)$$

These are the identities for our reparametrization invariant Lagrangian that follow from the second Noether theorem ([17]). The proof of Theorem 2.30 establishes the equivalence between the Noether identities (30) and the vanishing of the constant of motion \mathcal{J}_{X_τ} corresponding to every $X_\tau \in \text{diff}_+ \mathbf{R}$.

2.5. Arclength parametrization. Denote by $\langle x, x' \rangle$ the Euclidean scalar product and by $|x| = \sqrt{\langle x, x \rangle}$, the corresponding norm in \mathbf{R}^n . For a curve $c : I \rightarrow \mathbf{R}^n : t \mapsto x(t)$, where $I = [t_0, t_1]$, the arclength of the section of c from t_0 to t is

$$s(t) = \int_{t_0}^t |\dot{x}(t')| dt'. \quad (31)$$

In geometric problems it is often convenient to parametrize a curve in terms of its arclength. If t is the arclength of c , then along c

$$\langle \dot{x}, \dot{x} \rangle = |\dot{x}|^2 = 1, \quad (32)$$

$$\langle \dot{x}, \ddot{x} \rangle = 0, \quad (33)$$

$$\langle \dot{x}, \ddot{\ddot{x}} \rangle + \langle \ddot{x}, \ddot{x} \rangle = 0, \quad (34)$$

$$\langle \dot{x}, \ddot{\ddot{x}} \rangle + 3 \langle \ddot{x}, \ddot{x} \rangle = 0. \quad (35)$$

These equations determine submanifolds M^1, M^2, M^3 and M^4 of J^1, J^2, J^3 and J^4 , respectively.

PROPOSITION 2.32. *Let $X = \tau \frac{\partial}{\partial t}$ be a vector field on the configuration space Q . The necessary and sufficient condition for its prolongations X^1, X^2, X^3 and X^4 to be tangent to M^1, M^2, M^3 and M^4 , respectively, is that the restriction of τ to M^1, M^2, M^3 and M^4 , respectively, is constant.*

Proof. The vector field

$$X^1 = -\dot{\tau} \dot{x}^i \frac{\partial}{\partial \dot{x}^i} + \tau \frac{\partial}{\partial t}$$

on J^1 , generating the reparametrization transformation, differentiating $\langle \dot{x}, \dot{x} \rangle$ gives

$$\left(-\dot{\tau} \dot{x}^i \frac{\partial}{\partial \dot{x}^i} + \tau \frac{\partial}{\partial t} \right) \langle \dot{x}, \dot{x} \rangle = -\dot{\tau} \langle \dot{x}, \dot{x} \rangle.$$

Thus, X^1 is tangent to M^1 if and only if $\dot{\tau} = 0$. Moreover,

$$X^1 \langle \dot{x}, \dot{x} \rangle \Big|_{M^1} = -\dot{\tau}.$$

Next,

$$\begin{aligned} X^2 \langle \dot{x}, \ddot{x} \rangle &= \left(\tau \frac{\partial}{\partial t} - \dot{\tau} \dot{x}^i \frac{\partial}{\partial \dot{x}^i} - (2\ddot{x}_i \dot{\tau} + \dot{x}_i \ddot{\tau}) \frac{\partial}{\partial \ddot{x}^i} \right) \langle \dot{x}, \ddot{x} \rangle \\ &= -3\dot{\tau} \langle \dot{x}, \ddot{x} \rangle - \ddot{\tau} \langle \dot{x}, \dot{x} \rangle, \end{aligned}$$

and

$$(X^2 \langle \dot{x}, \ddot{x} \rangle) \Big|_{M^2} = -\ddot{\tau}.$$

Further,

$$\begin{aligned} &X^3 (\langle \dot{x}, \ddot{x} \rangle + \langle \ddot{x}, \ddot{x} \rangle) = \\ &= \left(\tau \frac{\partial}{\partial t} - \dot{\tau} \dot{x}^i \frac{\partial}{\partial \dot{x}^i} - (2\ddot{x}_i \dot{\tau} + \dot{x}_i \ddot{\tau}) \frac{\partial}{\partial \ddot{x}^i} - (3\ddot{x}_i \dot{\tau} + 3\ddot{x}_i \ddot{\tau} + \dot{x}_i \ddot{\tau}) \frac{\partial}{\partial \ddot{x}^i} \right) (\langle \dot{x}, \ddot{x} \rangle + \langle \ddot{x}, \ddot{x} \rangle) \\ &= -4\dot{\tau} \langle \dot{x}, \ddot{x} \rangle - 3\ddot{\tau} \langle \dot{x}, \ddot{x} \rangle - \ddot{\tau} \langle \dot{x}, \dot{x} \rangle - 4\dot{\tau} \langle \ddot{x}, \ddot{x} \rangle - 2\ddot{\tau} \langle \ddot{x}, \ddot{x} \rangle, \end{aligned}$$

and

$$X^3 (\langle \dot{x}, \ddot{x} \rangle + \langle \ddot{x}, \ddot{x} \rangle) \Big|_{M^3} = -\ddot{\tau}.$$

Finally,

$$\begin{aligned} &X^4 (\langle \dot{x}, x^{(4)} \rangle + 3\langle \ddot{x}, \ddot{x} \rangle) \\ &= \left(\tau \frac{\partial}{\partial t} - \dot{\tau} \dot{x}^i \frac{\partial}{\partial \dot{x}^i} - (2\ddot{x}_i \dot{\tau} + \dot{x}_i \ddot{\tau}) \frac{\partial}{\partial \ddot{x}^i} - (3\ddot{x}_i \dot{\tau} + 3\ddot{x}_i \ddot{\tau} + \dot{x}_i \ddot{\tau}) \frac{\partial}{\partial \ddot{x}^i} \right) (\langle \dot{x}, x^{(4)} \rangle + 3\langle \ddot{x}, \ddot{x} \rangle) + \\ &\quad - (4x^{(4)} \dot{\tau} + 6\ddot{x} \ddot{\tau} + 4\ddot{x} \ddot{\tau} + \dot{x} \tau^{(4)}) \frac{\partial}{\partial x^{(4)}} (\langle \dot{x}, x^{(4)} \rangle + 3\langle \ddot{x}, \ddot{x} \rangle) \\ &= -5\dot{\tau} \langle \dot{x}, x^{(4)} \rangle - 9\ddot{\tau} \langle \dot{x}, \ddot{x} \rangle - 9\ddot{\tau} \langle \ddot{x}, \ddot{x} \rangle - 15\dot{\tau} \langle \ddot{x}, \ddot{x} \rangle - 7\ddot{\tau} \langle \dot{x}, \ddot{x} \rangle - \tau^{(4)} \langle \dot{x}, \dot{x} \rangle. \end{aligned}$$

Therefore, $X^4 (\langle \dot{x}, x^{(4)} \rangle + 3\langle \ddot{x}, \ddot{x} \rangle) = 0$ if $\dot{\tau} = 0$ and

$$X^4 (\langle \dot{x}, x^{(4)} \rangle + 3\langle \ddot{x}, \ddot{x} \rangle) \Big|_{M^4} = -\tau^{(4)}.$$

q.e.d.

Suppose a local section σ of Q with domain $I \subset \mathbf{R}$ and with $j^1\sigma(I)$ not in M^1 , satisfies $\dot{x}(t) \neq 0$ for all $t \in I$.

LEMMA 2.33. *There exists $\varphi \in \text{Diff}_+ \mathbf{R}$ such that*

$$\frac{d\varphi}{dt} = |\dot{x}(t)|$$

for all $t \in I$.

Proof. This follows from the fundamental theorem of the calculus. q.e.d.

Then,

$$\frac{dx}{d\varphi} = \frac{dx}{dt} \frac{dt}{d\varphi} = \frac{dx}{dt} \frac{1}{|\dot{x}(t)|} = \frac{\dot{x}(t)}{|\dot{x}(t)|},$$

and it follows that

$$\left| \frac{dx}{d\varphi} \right| = 1.$$

Thus the new parametrization gives rise to a section $\varphi^*\sigma$ with its first jet in M^1 . Similarly, the k -jet of $\varphi^*\sigma$ is in M^k .

2.6. Hamiltonian formulation. The Liouville form on the cotangent bundle T^*J^1 with variables $(t, x, \dot{x}, p_t, p_x, p_{\dot{x}})$ is

$$\theta = p_t dt + p_x dx + p_{\dot{x}} d\dot{x}. \quad (36)$$

The exterior derivative

$$\omega = d\theta \quad (37)$$

is the canonical symplectic form of T^*J^1 .

LEMMA 2.34. *The action*

$$\text{Diff}_+ \mathbf{R} \times J^1 \rightarrow J^1 : (\varphi, (t, x, \dot{x})) \mapsto \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)} \right)$$

lifts to an action

$$\begin{aligned} \text{Diff}_+ \mathbf{R} \times T^*J^1 &\rightarrow T^*J^1 : \\ (\varphi, (t, x, \dot{x}, p_t, p_x, p_{\dot{x}})) &\mapsto \left(\varphi(t), x, \dot{\varphi}(t)^{-1} \dot{x}, p_t \dot{\varphi}(t)^{-1} + \langle p_{\dot{x}}, \dot{x} \rangle \dot{\varphi}(t)^{-2} \ddot{\varphi}(t), p_x, \dot{\varphi}(t) p_{\dot{x}} \right). \end{aligned} \quad (38)$$

The lifted action (38) is Hamiltonian with momentum map $\mathcal{J} : T^*J^1 \rightarrow \text{diff}_+ \mathbf{R}^*$ such that, for $X = \tau(t)\partial_t \in \text{diff}_+ \mathbf{R}$,

$$\mathcal{J}_X(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) = \tau(t)p_t - \dot{\tau}(t) \langle p_{\dot{x}}, \dot{x} \rangle. \quad (39)$$

Proof. The action of $\text{Diff}_+ \mathbf{R}$ on J^1 is given by

$$\text{Diff}_+ \mathbf{R} \times J^1 \rightarrow J^1 : (\varphi, (t, x, \dot{x})) \mapsto (t', x', \dot{x}') = \left(\varphi(t), x, \dot{\varphi}(t)^{-1} \dot{x} \right),$$

(see equation (26).) The lifted action takes $(p_t, p_x, p_{\dot{x}})$ to $(p'_t, p'_x, p'_{\dot{x}})$ such that

$$p'_t dt' + p'_x dx' + p'_{\dot{x}} d\dot{x}' = p_t dt + p_x dx + p_{\dot{x}} d\dot{x}.$$

But $dt' = \dot{\varphi}(t)dt$, $dx' = dx$ and $d\dot{x}' = \dot{\varphi}(t)^{-1}d\dot{x} - \dot{\varphi}(t)^{-2}\ddot{\varphi}(t)dt$. Hence,

$$\begin{aligned} p_t' dt' + p_x' dx' + p_{\dot{x}}' d\dot{x}' &= p_t' \dot{\varphi}(t)dt + p_x dx + p_{\dot{x}}' \dot{\varphi}(t)^{-1}d\dot{x} - \langle p_{\dot{x}}', \dot{x} \rangle \dot{\varphi}(t)^{-2}\ddot{\varphi}(t)dt \\ &= (p_t' \dot{\varphi}(t) - \langle p_{\dot{x}}', \dot{x} \rangle \dot{\varphi}(t)^{-2}\ddot{\varphi}(t))dt + p_x dx + p_{\dot{x}}' \dot{\varphi}(t)^{-1}d\dot{x} \\ &= p_t dt + p_x dx + p_{\dot{x}} d\dot{x}. \end{aligned}$$

Hence,

$$\begin{aligned} p_t' \dot{\varphi}(t) - \langle p_{\dot{x}}', \dot{x} \rangle \dot{\varphi}(t)^{-2}\ddot{\varphi}(t) &= p_t, \\ p_{x'} &= p_x, \\ p_{\dot{x}}' \dot{\varphi}(t)^{-1} &= p_{\dot{x}}. \end{aligned}$$

Therefore,

$$\begin{aligned} p_t' &= p_t \dot{\varphi}(t)^{-1} + \langle p_{\dot{x}}, \dot{x} \rangle \dot{\varphi}(t)^{-2}\ddot{\varphi}(t), \\ p_x' &= p_x, \\ p_{\dot{x}}' &= \dot{\varphi}(t)p_{\dot{x}}. \end{aligned}$$

The lifted action on the cotangent bundle is Hamiltonian, and the value of the momentum map on an element of the Lie algebra is given by the evaluation of the Liouville form on the vector field generating the action of the one-parameter group corresponding to this element of the Lie algebra. Hence, for $X = \tau(t)\partial_t \in \text{diff}_+ \mathbf{R}$,

$$\begin{aligned} \mathcal{J}_X(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) &= \langle \mathcal{J}, X \rangle(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) = \langle \theta, X^1 \rangle(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \\ &= \left\langle p_t dt + p_x dx + p_{\dot{x}} d\dot{x}, \tau \frac{\partial}{\partial t} - \dot{x} \tau \frac{\partial}{\partial \dot{x}} \right\rangle \\ &= \tau(t)p_t - \dot{\tau}(t) \langle p_{\dot{x}}, \dot{x} \rangle. \end{aligned}$$

q.e.d.

DEFINITION 2.35. The Legendre-Ostrogradski transformation

$$\mathcal{L} : J^3 \rightarrow T^*J^1 : (t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \mapsto (t, x, \dot{x}, p_t, p_x, p_{\dot{x}}), \quad (40)$$

is given by

$$\begin{aligned} p_t &= -H = -\dot{x}p_x - \ddot{x}p_{\dot{x}} + L \\ p_{\dot{x}} &= \frac{\partial L}{\partial \ddot{x}}, \\ p_x &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} p_{\dot{x}}. \end{aligned}$$

The Legendre transformation was extended by Ostrogradski in [18]. However, for brevity, from now on we shall just refer to it as the Legendre transformation.

Clearly, \mathcal{L} is a smooth map of J^3 into T^*J^1 . If \mathcal{L} is a diffeomorphism, Ostrogradski's approach leads to a regular time-dependent Hamiltonian theory with the Hamiltonian

$$H = -p_t = \dot{x}p_x + \ddot{x}p_{\dot{x}} - L.$$

In geometric problems, the Lagrangian is often reparametrization invariant. In this case L does not depend on t and the range of the Legendre transformation is restricted by the equations (see (2.30))

$$H = 0 \quad \text{and} \quad \langle p_{\dot{x}}, \dot{x} \rangle = 0.$$

THEOREM 2.36. *If the Lagrange form Ldt is invariant under the action of $\text{Diff}_+ \mathbf{R}$, then the Legendre transformation \mathcal{L} intertwines the actions of $\text{Diff}_+ \mathbf{R}$ on J^3 and on T^*J^1 .*

Proof. The map \mathcal{L} , defined by (40), intertwines with φ^3 , the induced action of $\text{Diff}_+ \mathbf{R}$ on J^3 , if and only if

$$\tilde{\varphi}^1 \circ \mathcal{L}(t, x, \dot{x}, \ddot{x}, \ddot{x}) = \mathcal{L} \circ \varphi^3(t, x, \dot{x}, \ddot{x}, \ddot{x}), \quad (41)$$

for every $(t, x, \dot{x}, \ddot{x}, \ddot{x}) \in J^3$. Here the map $\tilde{\varphi}^1: T^*J^1 \rightarrow T^*J^1$ is the lifted action of φ^1 to the cotangent bundle T^*J^1 , defined by (38).

Let us first compute the right hand side of (41). Let $L': J^2 \rightarrow \mathbf{R}$ be a function on J^2 such that $(\varphi^2)^*L' = L$. This function is an induced Lagrangian by the $\text{Diff}_+ \mathbf{R}$ -action. Based on (40), the map $\mathcal{L} \circ \varphi^3$ can be calculated by:

$$\begin{aligned} (t, x, \dot{x}, \ddot{x}, \ddot{x}) &\mapsto (t', x, x', p'_t, p'_{\dot{x}}, p'_x) \\ t' &= \varphi(t) \\ x' &= \frac{\dot{x}}{\dot{\varphi}} \\ p'_t &= -\langle p'_{\dot{x}}, x' \rangle - \langle p'_x, x'' \rangle + L' \\ p'_{\dot{x}} &= \frac{\partial L'}{\partial x''} \\ p'_x &= \frac{\partial L'}{\partial x'} - \frac{d}{dt'}(p'_{\dot{x}}) \\ x'' &= \frac{\ddot{x}}{\dot{\varphi}^2} - \frac{\ddot{\varphi}\dot{x}}{\dot{\varphi}^3}. \end{aligned}$$

In order to calculate the partial derivatives of L' , we use the assumption that the Lagrange form Ldt is invariant under the $\text{Diff}_+ \mathbf{R}$ -action, i.e.,

$$(\varphi^2)^*(L'dt') = (L' \circ \varphi^2)\dot{\varphi}dt = Ldt \implies L = \dot{\varphi}(L' \circ \varphi^2).$$

Therefore, the partial derivatives of L' with respect to x, x' and x'' are calculated as

$$\begin{aligned} \frac{\partial L}{\partial x} &= \dot{\varphi} \frac{\partial L'}{\partial x} \implies \frac{\partial L'}{\partial x} = \frac{1}{\dot{\varphi}} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial \ddot{x}} &= \dot{\varphi} \left(\frac{\partial L'}{\partial x''} \frac{\partial x''}{\partial \ddot{x}} \right) = \frac{1}{\dot{\varphi}} \frac{\partial L'}{\partial x''} \implies \frac{\partial L'}{\partial x''} = \dot{\varphi} \frac{\partial L}{\partial \ddot{x}} \\ \frac{\partial L}{\partial \dot{x}} &= \dot{\varphi} \left(\frac{\partial L'}{\partial x'} \frac{\partial x'}{\partial \dot{x}} + \frac{\partial L'}{\partial x''} \frac{\partial x''}{\partial \dot{x}} \right) = \frac{\partial L'}{\partial x'} - \frac{\ddot{\varphi}}{\dot{\varphi}^2} \frac{\partial L'}{\partial x''} \implies \frac{\partial L'}{\partial x'} = \frac{\partial L}{\partial \dot{x}} + \frac{\ddot{\varphi}}{\dot{\varphi}} \frac{\partial L}{\partial \ddot{x}} \end{aligned}$$

Since Ldt is parametrization invariant, by theorem (2.30), L is time invariant, $p_t = -H = 0$ and $\langle p_{\dot{x}}, \dot{x} \rangle = 0$. As the result, we show that the partial derivative of L'

with respect to t' is equal to zero:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial t} = \ddot{\varphi} L' + \dot{\varphi} \left(\frac{\partial L'}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial L'}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial L'}{\partial x''} \frac{\partial x''}{\partial t} \right) \\
&= \ddot{\varphi} \left(L' + \frac{1}{\dot{\varphi}} \left[- \left\langle \frac{\partial L'}{\partial x'}, \dot{x} \right\rangle - \frac{2}{\dot{\varphi}} \left\langle \frac{\partial L'}{\partial x''}, \ddot{x} \right\rangle + \frac{3\ddot{\varphi}}{\dot{\varphi}^2} \left\langle \frac{\partial L'}{\partial x''}, \dot{x} \right\rangle \right] \right) + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} - \frac{\ddot{\varphi}}{\dot{\varphi}^2} \left\langle \frac{\partial L'}{\partial x''}, \dot{x} \right\rangle \\
&= \frac{\ddot{\varphi}}{\dot{\varphi}} \left(L - \left\langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \right\rangle - \frac{\ddot{\varphi}}{\dot{\varphi}} \langle p_{\dot{x}}, \dot{x} \rangle - 2 \langle p_{\ddot{x}}, \ddot{x} \rangle + \frac{3\ddot{\varphi}}{\dot{\varphi}^2} \langle p_{\dot{x}}, \dot{x} \rangle \right) + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} - \frac{\ddot{\varphi}}{\dot{\varphi}} \langle p_{\dot{x}}, \dot{x} \rangle \\
&= \frac{\ddot{\varphi}}{\dot{\varphi}} \left(L - \left\langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \right\rangle - 2 \langle p_{\ddot{x}}, \ddot{x} \rangle \right) + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} \\
&= \frac{\ddot{\varphi}}{\dot{\varphi}} \left(L - \left\langle \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} (p_{\dot{x}}), \dot{x} \right\rangle - \frac{d}{dt} \langle p_{\ddot{x}}, \dot{x} \rangle + \langle p_{\dot{x}}, \ddot{x} \rangle - 2 \langle p_{\ddot{x}}, \ddot{x} \rangle \right) + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} \\
&= \frac{\ddot{\varphi}}{\dot{\varphi}} (L - \langle p_x, \dot{x} \rangle - \langle p_{\ddot{x}}, \ddot{x} \rangle) + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} = \frac{\ddot{\varphi}}{\dot{\varphi}} p_t + \dot{\varphi}^2 \frac{\partial L'}{\partial t'} = \dot{\varphi}^2 \frac{\partial L'}{\partial t'} \implies \frac{\partial L'}{\partial t'} = 0.
\end{aligned}$$

Now, we can calculate the right hand side of (41) in terms of elements of J^3 :

$$\begin{aligned}
p'_{\dot{x}} &= \frac{\partial L'}{\partial x''} = \dot{\varphi} \frac{\partial L}{\partial \ddot{x}} = \dot{\varphi} p_{\ddot{x}} \\
p'_x &= \frac{\partial L'}{\partial x'} - \frac{d}{dt'} (p'_{\dot{x}}) = \frac{\partial L}{\partial \dot{x}} + \frac{\ddot{\varphi}}{\dot{\varphi}} \frac{\partial L}{\partial \ddot{x}} - \frac{1}{\dot{\varphi}} \frac{d}{dt} \left(\dot{\varphi} \frac{\partial L}{\partial \ddot{x}} \right) = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} (p_{\dot{x}}) = p_x \\
p'_t &= - \langle p'_x, x' \rangle - \langle p'_{\dot{x}}, x'' \rangle + L' = - \left\langle p_x, \frac{\dot{x}}{\dot{\varphi}} \right\rangle - \left\langle \dot{\varphi} p_{\ddot{x}}, \frac{\ddot{x}}{\dot{\varphi}^2} - \frac{\ddot{\varphi} \dot{x}}{\dot{\varphi}^3} \right\rangle + \frac{L}{\dot{\varphi}} \\
&= \frac{1}{\dot{\varphi}} (- \langle p_x, \dot{x} \rangle - \langle p_{\ddot{x}}, \ddot{x} \rangle + L) = \frac{1}{\dot{\varphi}} p_t = 0,
\end{aligned}$$

where p_x and $p_{\ddot{x}}$ are considered as functions on J^3 , defined by:

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right), \\
p_{\ddot{x}} &= \frac{\partial L}{\partial \ddot{x}}.
\end{aligned}$$

In the following, we calculate the left hand side of (41). Since $L dt$ is $\text{Diff}_+ \mathbf{R}$ -invariant, and based on (38):

$$\tilde{\varphi}^1 \circ \mathcal{L}(t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) = \tilde{\varphi}^1(t, x, \dot{x}, 0, p_x, p_{\ddot{x}}) = (\varphi(t), x, \dot{x}/\dot{\varphi}, 0, p_x, \dot{\varphi} p_{\ddot{x}}).$$

Therefore, for $\text{Diff}_+ \mathbf{R}$ -invariant Lagrange forms the relation (41) holds. This completes the proof of the theorem. q.e.d.

COROLLARY 2.37. *If the Lagrange form $L dt$ is $\text{Diff}_+ \mathbf{R}$ -invariant, then the Cartan form Θ is $\text{Diff}_+ \mathbf{R}$ -invariant.*

Proof. Since $L dt$ is $\text{Diff}_+ \mathbf{R}$ -invariant, theorem (2.36) implies that for $\varphi \in \text{Diff}_+ \mathbf{R}$, $\varphi^{3*} \mathcal{L}^* = \mathcal{L}^* \tilde{\varphi}^{1*}$, where $\tilde{\varphi}^1$ is the lift of φ^1 to the cotangent bundle $T^* J^1$. But, $\Theta = \mathcal{L}^* \theta$, and the Liouville form θ is invariant under the the lifted action $\tilde{\varphi}^1$. Therefore,

$$\varphi^{3*} \Theta = \varphi^{3*} \mathcal{L}^* \theta = \mathcal{L}^* \tilde{\varphi}^{1*} \theta = \mathcal{L}^* \theta = \Theta.$$

q.e.d.

COROLLARY 2.38. *The range of the Legendre transformation $\mathcal{L} : J^3 \rightarrow T^*J^1$ is contained in the zero set of the momentum map $\mathcal{J} : T^*J^1 \rightarrow \text{diff}_+ \mathbf{R}^*$. That is,*

$$\mathcal{L}(J^3) \subseteq \mathcal{J}^{-1}(0).$$

3. CLASSICAL ELASTICA

3.1. **The variational equations.** The elastica functional is given by

$$A[\sigma] = \int_{t_0}^{t_1} \kappa^2 |\dot{x}| dt,$$

where

$$\sigma : [t_0, t_1] \rightarrow [t_0, t_1] \times \mathbf{R}^3 : t \mapsto (t, x(t))$$

corresponds to a curve $\sigma : t \mapsto x(t)$ in \mathbf{R}^3 , and κ is the curvature of σ . Since the curvature of the curve depends on its second derivatives, this is naturally a second order variational problem. As the curvature in Cartesian coordinates $x = (x^1, x^2, x^3) \in \mathbf{R}^3$ is

$$\kappa^2 = \frac{|\ddot{x}|^2}{|\dot{x}|^4} - \frac{\langle \dot{x}, \ddot{x} \rangle^2}{|\dot{x}|^6},$$

the elastica Lagrangian is

$$L(x, \dot{x}, \ddot{x}) = \frac{|\ddot{x}|^2}{|\dot{x}|^3} - \frac{\langle \dot{x}, \ddot{x} \rangle^2}{|\dot{x}|^5}. \quad (42)$$

It is defined on $\{(t, x, \dot{x}, \ddot{x}) \in J^2 \mid \dot{x} \neq 0\}$.

PROPOSITION 3.1. *The elastica Lagrangian (42) is invariant under translations and rotations in \mathbf{R}^3 and is independent of parametrization.*

Proof. The expression (42) for L is independent of x and depends only on Euclidean scalar products of \dot{x} and \ddot{x} . Hence, L is invariant under translations and rotations. Moreover, the curvature κ of a curve is independent of its parametrization, and $|\dot{x}| dt = ds$ is the element of arclength. Therefore, $L dt = \kappa^2 ds$ is independent of parametrization. q.e.d.

For elastica, Ostrogradski's momenta are

$$p_{\ddot{x}} = 2 \frac{\ddot{x}}{|\dot{x}|^3} - 2 \frac{\langle \dot{x}, \ddot{x} \rangle \dot{x}}{|\dot{x}|^5},$$

and

$$p_x = -\frac{2}{\langle \dot{x}, \dot{x} \rangle^{5/2}} (\langle \dot{x}, \dot{x} \rangle \ddot{x} - \langle \dot{x}, \ddot{x} \rangle \dot{x}) - \frac{\langle \ddot{x}, \ddot{x} \rangle \dot{x}}{\langle \dot{x}, \dot{x} \rangle^{5/2}} + 6 \frac{\langle \dot{x}, \ddot{x} \rangle \ddot{x}}{\langle \dot{x}, \dot{x} \rangle^{5/2}} - 5 \frac{\langle \dot{x}, \ddot{x} \rangle^2 \dot{x}}{\langle \dot{x}, \dot{x} \rangle^{7/2}}.$$

The Euler-Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} = 0$$

can be written in the form

$$\frac{\partial L}{\partial x} - \frac{d}{dt} p_x = 0.$$

Since the Lagrangian is parameter-independent, the Euler-Lagrange equations are necessarily degenerate in that they do not determine the fourth derivative \ddot{x} uniquely. Let

$$\ddot{x}^{\parallel} = \frac{\langle \ddot{x}, \dot{x} \rangle}{|\dot{x}|^2} \dot{x} \quad \text{and} \quad \ddot{x}^{\perp} = \ddot{x} - \ddot{x}^{\parallel}$$

denote the components of \ddot{x} that are parallel and perpendicular to \dot{x} , respectively. The Euler-Lagrange equations written in terms of this decomposition are

$$\ddot{x}^{\perp} = 6|\dot{x}|^2 \langle \dot{x}, \ddot{x} \rangle \ddot{x} + 4 \langle \dot{x}, \ddot{x} \rangle \ddot{x} + \frac{5}{2} |\ddot{x}|^2 \ddot{x} - 10 \langle \dot{x}, \ddot{x} \rangle \langle \dot{x}, \ddot{x} \rangle - \frac{5 |\ddot{x}|^2 \langle \dot{x}, \ddot{x} \rangle}{2 |\dot{x}|^2} \dot{x} + \frac{35 \langle \dot{x}, \ddot{x} \rangle^3}{4 |\dot{x}|^6} \dot{x} \quad (43)$$

They determine \ddot{x}^{\perp} , but leave the component \ddot{x}^{\parallel} undetermined. On the other hand, the parametrization-invariance of the problem allows us to use parametrization by the arclength. In the following, assume that t is the arclength parameter of the curve. Therefore

$$|\dot{x}|^2 = \langle \dot{x}, \dot{x} \rangle = 1, \quad (44)$$

and, by differentiation

$$\langle \dot{x}, \ddot{x} \rangle = 0, \quad (45)$$

$$\langle \dot{x}, \ddot{x} \rangle + \langle \ddot{x}, \ddot{x} \rangle = 0, \quad (46)$$

$$\langle \dot{x}, \ddot{x} \rangle + 3 \langle \ddot{x}, \ddot{x} \rangle = 0, \quad (47)$$

as well. Substitution into (43) and (47) yields

$$\ddot{x}^{\perp} = -\frac{3}{2} |\ddot{x}|^2 \ddot{x} \quad \text{and} \quad \ddot{x}^{\parallel} = -3 \langle \ddot{x}, \ddot{x} \rangle \dot{x}. \quad (48)$$

These equations determine the elastica completely. In other words, the choice of a parametrization determines an equation for the component \ddot{x}^{\parallel} .

REMARK 3.2. This yields an equation of the form

$$\ddot{x} = f(x, \dot{x}, \ddot{x}, \ddot{x})$$

to which theorems in differential equations apply that guarantee the local existence and uniqueness of solutions.

3.2. The Frenet frame. The elastica equations (48) are conveniently studied in the moving frame (T, N, B) , where $T = \dot{x}$ is the unit tangent vector, N the normal vector and B the binormal vector of the curve $t \mapsto x(t)$. The Frenet equations are

$$\dot{T} = \kappa N, \quad (49)$$

$$\dot{N} = -\kappa T + \tau B, \quad (50)$$

$$\dot{B} = -\tau N, \quad (51)$$

with $\kappa = |\ddot{x}|$ the curvature and τ the torsion of the curve. In order to relate the torsion τ to the derivative variables, observe that

$$\ddot{x} = \dot{\kappa} N + \kappa \dot{N} = \dot{\kappa} N - \kappa^2 \dot{x} + \kappa \tau B, \quad (52)$$

which implies that, if $\kappa \neq 0$,

$$\tau = \kappa^{-1} \langle B, \ddot{x} \rangle = \kappa^{-1} \langle T \times N, \ddot{x} \rangle = \kappa^{-2} \langle \dot{x} \times \ddot{x}, \ddot{x} \rangle. \quad (53)$$

Differentiating (52) and the Frenet equations imply

$$\ddot{x} = -3\kappa\dot{\kappa}T + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)N + (2\dot{\kappa}\tau + \kappa\dot{\tau})B. \quad (54)$$

This, together with equation (48) implies that

$$2\dot{\kappa}\tau + \kappa\dot{\tau} = 0, \quad (55)$$

$$2\ddot{\kappa} + \kappa^3 - 2\kappa\tau^2 = 0. \quad (56)$$

Equation (48) does not lead to any new condition because $\kappa = |\ddot{x}|$ implies that $\langle \ddot{x}, \ddot{x} \rangle = \kappa\dot{\kappa}$. Equation (55) can be immediately integrated to yield

$$\kappa^2\tau = c, \quad c \text{ a constant.} \quad (57)$$

If $\kappa \neq 0$, substituting $\tau = \frac{c}{\kappa^2}$ into equation (56) and integrating gives

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \frac{c^2}{\kappa^2} = \text{constant.} \quad (58)$$

Integration of equation (58) determines completely the functions $\kappa(t)$ and $\tau(t)$ in terms of the initial data $\kappa(t_0)$, $\dot{\kappa}(t_0)$ and $\tau(t_0)$. Thus, in order to find the solution $t \mapsto x(t)$, it suffices to integrate Frenet's equations assuming that the curvature κ and the torsion τ are known functions of t . This can be achieved using the conservation laws for elastica.

3.3. Conserved momenta. Since the Lagrange form for elastica is invariant under translations, it follows that the linear momentum $p = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right)$ is conserved. In the arclength parametrization

$$-2(\ddot{x} - \langle \dot{x}, \ddot{x} \rangle \dot{x}) - \langle \ddot{x}, \ddot{x} \rangle \dot{x} = p = \text{constant.} \quad (59)$$

The arclength parametrization implies

$$p = -2\ddot{x} - 3 \langle \ddot{x}, \ddot{x} \rangle \dot{x}. \quad (60)$$

Similarly, rotational invariance of the Lagrange form implies that the angular momentum

$$\mathcal{J}_{X_{ij}} = x^i p_{x^j} - x^j p_{x^i} + \dot{x}^i p_{\dot{x}^j} - \dot{x}^j p_{\dot{x}^i}$$

is conserved (see example 2.19.) Setting $l = (l_1, l_2, l_3)$, where

$$l_i = \epsilon_{ijk} \mathcal{J}_{X_{jk}},$$

gives

$$l = x \times p_x + \dot{x} \times p_{\dot{x}}. \quad (61)$$

The expression (42) for the elastica Lagrangian in arbitrary parametrization yields, in the arclength parametrization

$$p_{\dot{x}} = 2\ddot{x} - \dot{x},$$

which implies that

$$l = x \times p + 2\dot{x} \times \ddot{x}. \quad (62)$$

PROPOSITION 3.3. *The conserved momenta in the moving frame (T, N, B) are*

$$p = -\kappa^2 T - 2\dot{\kappa}N - 2\kappa\tau B, \quad (63)$$

$$l = x \times p + 2\kappa B. \quad (64)$$

Proof. Equation (52) implies that

$$p = -2(\dot{\kappa}N - \kappa^2 T + \kappa\tau B) - 3\kappa^2 T = -\kappa^2 T - 2\dot{\kappa}N - 2\kappa\tau B.$$

while (49) and (51) yield

$$\dot{x} \times \ddot{x} = T \times (\kappa N) = \kappa(T \times N) = \kappa B. \quad (65)$$

q.e.d.

PROPOSITION 3.4. *Scalar equations for the curvature and torsion (57) and (58) can be rewritten in the form*

$$\kappa^2 \tau = -\frac{1}{4} \langle l, p \rangle, \quad (66)$$

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \frac{\langle l, p \rangle^2}{16\kappa^2} = \frac{1}{4}|p|^2. \quad (67)$$

Proof. Equations (63) and (64) imply

$$\langle l, p \rangle = 2\kappa \langle B, p \rangle = 2\kappa \langle B, -\kappa^2 T - 2\dot{\kappa}N - 2\kappa\tau B \rangle = -4\kappa^2 \tau.$$

Equation (63) implies

$$\begin{aligned} |p|^2 &= \kappa^4 + 4\dot{\kappa}^2 + 4\kappa^2 \tau^2 = 4\left(\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \frac{\kappa^4 \tau^2}{\kappa^2}\right) \\ &= 4\left(\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \frac{\langle l, p \rangle^2}{16\kappa^2}\right). \end{aligned}$$

q.e.d.

Suppose that κ and τ as known as functions of t (the differential equations imply that they may be expressed as elliptic functions.) It remains to show how the integration of the Frenet equations is aided by the conservation laws

$$\begin{aligned} p &= -2\dot{\kappa}N - 2\kappa\tau B - \kappa^2 T, \\ l &= x \times p + 2\kappa B. \end{aligned}$$

THEOREM 3.5. *The velocity of the elastica in the direction of the conserved momentum p is*

$$\langle \dot{x}, p \rangle = -\kappa^2. \quad (68)$$

Hence

$$\langle x, p \rangle = \langle x(t_0), p \rangle - \int_{t_0}^t \kappa^2 ds. \quad (69)$$

Proof. Taking the scalar product of l and p yields

$$\langle B, p \rangle = \frac{1}{2\kappa} \langle l, p \rangle.$$

Moreover, by taking the derivative of both sides of this equation and using the third Frenet equation,

$$\frac{d}{dt} \langle B, p \rangle = \langle \dot{B}, p \rangle = -\tau \langle N, p \rangle = \frac{d}{dt} \left(\frac{1}{2\kappa} \right) \langle l, p \rangle = \frac{-\dot{\kappa}}{2\kappa^2} \langle l, p \rangle.$$

Therefore if $\tau \neq 0$, by (66) we have

$$\langle N, p \rangle = \frac{\dot{\kappa}}{2\tau\kappa^2} \langle l, p \rangle = \frac{-2\dot{\kappa}}{\langle l, p \rangle} \langle l, p \rangle = -2\dot{\kappa}.$$

Finally, the second Frenet equation yields

$$\frac{d}{dt} \langle N, p \rangle = -\kappa \langle \dot{x}, p \rangle + \tau \langle B, p \rangle = -2\ddot{\kappa}.$$

Therefore if $\kappa \neq 0$, (56) implies

$$\langle \dot{x}, p \rangle = \frac{2\ddot{\kappa}}{\kappa} + \frac{\tau}{\kappa} \langle B, p \rangle = \frac{-\kappa^3 + 2\kappa\tau^2}{\kappa} + \frac{\tau}{2\kappa^2} \langle l, p \rangle = -\kappa^2 + 2\tau^2 + \frac{\tau}{2\kappa^2} (-4\kappa^2\tau) = -\kappa^2.$$

In particular,

$$\langle x, p \rangle = \langle x(t_0), p \rangle + \int_{t_0}^t \langle \dot{x}(s), p \rangle ds = \langle x(t_0), p \rangle - \int_{t_0}^t \kappa^2 ds.$$

q.e.d.

It remains to determine the component of the motion perpendicular to p . If \dot{x} is not parallel to p , then the vectors $\dot{x} \times p$ and $(\dot{x} \times p) \times p$ span the plane of directions perpendicular to p . Their lengths are

$$|\dot{x} \times p| = |-2\dot{\kappa}B + 2\kappa\tau N| = \sqrt{4\dot{\kappa}^2 + 4\kappa^2\tau^2} = \sqrt{|p|^2 - \kappa^4},$$

and

$$|(\dot{x} \times p) \times p| = \sqrt{|p|^4 - |p|^2 \kappa^4} = |p| \sqrt{|p|^2 - \kappa^4}.$$

Let D and E denote the unit vectors in the direction of $\dot{x} \times p$ and $(\dot{x} \times p) \times p$, respectively. Equations (63) and (64) give

$$D = \frac{\dot{x} \times p}{|\dot{x} \times p|} = \left(|p|^2 - \kappa^4 \right)^{-1/2} (-2\dot{\kappa}B + 2\kappa\tau N). \quad (70)$$

Similarly,

$$E = \frac{(\dot{x} \times p) \times p}{|(\dot{x} \times p) \times p|} = \frac{-(4\dot{\kappa}^2 + 4\kappa^2\tau^2)T + 2\kappa^2\dot{\kappa}N + 2\kappa^3\tau B}{|p| \left(|p|^2 - \kappa^4 \right)^{1/2}} \quad (71)$$

PROPOSITION 3.6. *The frame $(D(t), E(t))$ satisfies the equations*

$$\begin{aligned}\dot{D} &= -\frac{\langle l, p \rangle |p|}{2(|p|^2 - \kappa^4)} E, \\ \dot{E} &= \frac{\langle l, p \rangle |p|}{2(|p|^2 - \kappa^4)} D,\end{aligned}$$

where $\kappa^2 \tau = -\frac{1}{4} \langle l, p \rangle$.

Proof. Equations (63) and (64) give

$$\dot{x} \times p = -2\dot{k}B - 2\kappa\dot{B} = -2\dot{k}B + 2\kappa\tau N \quad (72)$$

and

$$(\dot{x} \times p) \times p = \langle p, p \rangle \dot{x} - \langle \dot{x}, p \rangle p = (4\dot{k}^2 + 4\kappa^2\tau^2)T - 2\kappa^2\dot{k}N - 2\kappa^3\tau B.$$

Differentiation yields

$$\begin{aligned}\frac{d}{dt}(\dot{x} \times p) &= \ddot{x} \times p = \kappa N \times (-2\dot{k}N - 2\kappa\tau B - \kappa^2 T) \\ &= -2\kappa^2\tau T + \kappa^3 B \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}\{(\dot{x} \times p) \times p + 2\kappa^2\dot{k}N + 2\kappa^3\tau B\} + \kappa^3 B \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p - \frac{4\kappa^4\dot{k}\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}N + \frac{4\kappa^2}{(4\dot{k}^2 + 4\kappa^2\tau^2)}\kappa^3 B \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(2\kappa\tau N - 2\dot{k}B) \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p),\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(\dot{x} \times p) \times p &= (\ddot{x} \times p) \times p \\ &= \left(-\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \right) \times p \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}((\dot{x} \times p) \times p) \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p \\ &= -\frac{2\kappa^2\tau}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(-\dot{x}|p|^2 + p \langle \dot{x}, p \rangle) \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p \\ &= \frac{2\kappa^2\tau|p|^2}{(4\dot{k}^2 + 4\kappa^2\tau^2)}\dot{x} \times p - \frac{2\kappa^3\dot{k}}{(4\dot{k}^2 + 4\kappa^2\tau^2)}(\dot{x} \times p) \times p.\end{aligned}$$

Now compute for the orthonormal frame

$$\{|\dot{x} \times p|^{-1} \dot{x} \times p, |(\dot{x} \times p) \times p|^{-1} (\dot{x} \times p) \times p\}.$$

Since

$$|\dot{x} \times p| = |-2\dot{k}b + 2\kappa\tau n| = \sqrt{4\dot{k}^2 + 4\kappa^2\tau^2} = \sqrt{|p|^2 - \kappa^4},$$

it follows that

$$\frac{d}{dt} |\dot{x} \times p|^{-1} = \frac{d}{dt} (|p|^2 - \kappa^4)^{-1/2} = -\frac{1}{2} (|p|^2 - \kappa^4)^{-3/2} (-4\kappa^3 \dot{\kappa}) = 2\kappa^3 \dot{\kappa} (|p|^2 - \kappa^4)^{-3/2}.$$

This further implies

$$\begin{aligned} \frac{d}{dt} (\dot{x} \times p) &= -\frac{2\kappa^2 \tau}{(4\dot{\kappa}^2 + 4\kappa^2 \tau^2)} (\dot{x} \times p) \times p - \frac{2\kappa^3 \dot{\kappa}}{(4\dot{\kappa}^2 + 4\kappa^2 \tau^2)} (\dot{x} \times p) \quad (73) \\ &= -\frac{2\kappa^2 \tau}{|p|^2 - \kappa^4} (\dot{x} \times p) \times p - \frac{2\kappa^3 \dot{\kappa}}{|p|^2 - \kappa^4} (\dot{x} \times p), \\ \frac{d}{dt} (\dot{x} \times p) \times p &= \frac{2\kappa^2 \tau |p|^2}{(4\dot{\kappa}^2 + 4\kappa^2 \tau^2)} \dot{x} \times p - \frac{2\kappa^3 \dot{\kappa}}{(4\dot{\kappa}^2 + 4\kappa^2 \tau^2)} (\dot{x} \times p) \times p \\ &= \frac{2\kappa^2 \tau |p|^2}{|p|^2 - \kappa^4} \dot{x} \times p - \frac{2\kappa^3 \dot{\kappa}}{|p|^2 - \kappa^4} (\dot{x} \times p) \times p. \end{aligned}$$

Similarly,

$$|(\dot{x} \times p) \times p|^2 = |-\dot{x} |p|^2 + p \langle \dot{x}, p \rangle|^2 = |p|^4 - |p|^2 \kappa^4,$$

and thus

$$\begin{aligned} \frac{d}{dt} |(\dot{x} \times p) \times p|^{-1} &= \frac{d}{dt} (|p|^4 - |p|^2 \kappa^4)^{-1/2} = |p|^{-1} \frac{d}{dt} (|p|^2 - \kappa^4)^{-1/2} \\ &= \frac{2\kappa^3 \dot{\kappa}}{|p|} (|p|^2 - \kappa^4)^{-3/2}. \end{aligned}$$

Therefore,

$$\frac{d}{dt} (|\dot{x} \times p|^{-1} \dot{x} \times p) = \frac{d}{dt} (|\dot{x} \times p|^{-1}) \dot{x} \times p + |\dot{x} \times p|^{-1} \frac{d}{dt} (\dot{x} \times p) \quad (74)$$

$$= -\frac{2\kappa^2 \tau}{(|p|^2 - \kappa^4)^{3/2}} (\dot{x} \times p) \times p \quad (75)$$

$$= -\frac{2\kappa^2 \tau |p|}{(|p|^2 - \kappa^4) |(\dot{x} \times p) \times p|} (\dot{x} \times p) \times p. \quad (76)$$

Similarly,

$$\begin{aligned} \frac{d}{dt} (|(\dot{x} \times p) \times p|^{-1} (\dot{x} \times p) \times p) &= \left(\frac{d}{dt} |(\dot{x} \times p) \times p|^{-1} \right) (\dot{x} \times p) \times p + \\ &\quad + |(\dot{x} \times p) \times p| \frac{d}{dt} ((\dot{x} \times p) \times p) \\ &= (|p|^2 - \kappa^4)^{-3/2} 2\kappa^2 \tau |p| \dot{x} \times p \\ &= \frac{2\kappa^2 \tau |p|}{(|p|^2 - \kappa^4)^{3/2}} \frac{|\dot{x} \times p|}{|\dot{x} \times p|} \dot{x} \times p \\ &= \frac{2\kappa^2 \tau |p|}{(|p|^2 - \kappa^4) |\dot{x} \times p|} \dot{x} \times p. \end{aligned}$$

Thus,

$$\begin{aligned}\dot{D} &= -\frac{2\kappa^2\tau|p|}{(|p|^2 - \kappa^4)}E, \\ \dot{E} &= \frac{2\kappa^2\tau|p|}{(|p|^2 - \kappa^4)}D,\end{aligned}$$

and the proof is finished since $\kappa^2\tau = -\frac{1}{4}\langle l, p \rangle$. q.e.d.

Define the curve of complex-valued vectors $Z(t)$ by

$$Z(t) = D(t) + iE(t). \quad (77)$$

Proposition 3.6 implies

$$\dot{Z} = \dot{D} + i\dot{E} = -i\frac{\langle l, p \rangle |p|}{2(|p|^2 - \kappa^4)}Z.$$

PROPOSITION 3.7. *Define*

$$\phi(t) = \langle l, p \rangle |p| \int_{t_0}^t \frac{1}{(|p|^2 - \kappa^4)} ds,$$

and set $Z_0 = Z(t_0) = D_0 + iE_0$, then

$$Z(t) = e^{-i\phi(t)} Z_0. \quad (78)$$

In particular,

$$D(t) = \cos \phi(t) D_0 + \sin \phi(t) E_0, \quad (79)$$

$$E(t) = -\sin \phi(t) D_0 + \cos \phi(t) E_0. \quad (80)$$

Proof. This follows immediately upon differentiating

$$\dot{Z} = \frac{d}{dt}Z = \frac{d}{dt}e^{-i\phi(t)}Z_0 = -i\dot{\phi}e^{-i\phi(t)}Z_0 = -i\frac{\langle l, p \rangle |p|}{2(|p|^2 - \kappa^4)}Z,$$

and $Z(t_0) = e^{-i\phi(t_0)}Z_0 = Z_0$. q.e.d.

Note that

$$(\dot{x} \times p) \times p = -|p|^2 \dot{x} + \langle \dot{x}, p \rangle p$$

implies that

$$\dot{x}_p^\perp := -|p|^{-2} (\dot{x} \times p) \times p$$

is the component of \dot{x} perpendicular to p .

THEOREM 3.8. *The time evolution of \dot{x}_p^\perp is*

$$\dot{x}_p^\perp(t) = -\frac{(|p|^2 - \kappa(t)^4)^{1/2}}{|p|} (-\sin \phi(t) D_0 + \cos \phi(t) E_0).$$

Hence, the component $t \mapsto x_0 + x_p^\perp(t)$ of the motion of the elastica in the plane perpendicular to p through $x_0 = x(t_0)$ is

$$x_p^\perp(t) = -|p|^{-2} (x_0 \times p) \times p - \frac{1}{|p|} \int_{t_0}^t (|p|^2 - \kappa(s)^4)^{1/2} (-\sin \phi(s) D_0 + \cos \phi(s) E_0) ds.$$

Proof.

$$\begin{aligned}
 \dot{x}_p^\perp &= \frac{|(\dot{x} \times p) \times p|}{|p|^2} E \\
 &= \frac{|p|(|p|^2 - \kappa^4)^{1/2}}{|p|^2} E \\
 &= \frac{(|p|^2 - \kappa^4)^{1/2}}{|p|} (-\sin \phi(t) D_0 + \cos \phi(t) E_0).
 \end{aligned}$$

q.e.d.

COROLLARY 3.9. *The elastica equations in the arclength parametrization*

$$\sigma : I \rightarrow R^n : t \mapsto x(t),$$

have a unique solution

$$x(t) = x_0 + \int_{t_0}^t \left(-\frac{\kappa(s)^2}{|p|^2} p - \frac{(|p|^2 - \kappa(s)^4)^{1/2}}{|p|} (-\sin \phi(s) D_0 + \cos \phi(s) E_0) \right) ds$$

for initial data in

$$M_0^3 = \{(t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \in M^3 \mid \kappa \neq 0, \tau \neq 0\}.$$

It remains to consider the special cases when $\kappa \neq 0$ and $\tau = 0$, and when $\kappa = 0$. Equation (66), $\kappa^2 \tau = -\frac{1}{4} \langle l, p \rangle$, shows that $\langle l, p \rangle = 0$. Hence, if either κ or τ vanishes at some point t_0 , then it vanishes for all t for which the solution exists.

- (1) If $\tau = 0$ and $\kappa \neq 0$, the Frenet equations are $\dot{T} = \kappa N$, $\dot{N} = -\kappa T$, $\dot{B} = 0$, and the conservation of the linear momentum p and the angular momentum l are

$$\begin{aligned}
 p &= -2\kappa N - \kappa^2 T, \\
 l &= x \times p + 2\kappa B.
 \end{aligned} \tag{81}$$

Thus, there is an additional conserved quantity,

$$B = \frac{1}{\kappa} \dot{x} \times \ddot{x}.$$

As the scalar product $\langle \dot{x}, p \rangle = -\kappa^2 \langle \dot{x}, \dot{x} \rangle = -\kappa^2$,

$$\langle x(t), p \rangle = \langle x(t_0), p \rangle - \int_{t_0}^t \kappa^2(s) ds.$$

Taking the cross product of equation (81) with B yields

$$-2\kappa N \times B - \kappa^2 T \times B = p \times B.$$

Since $N \times B = T$ and $T \times B = -N$,

$$-2\kappa T + \kappa^2 N = p \times B.$$

Therefore,

$$\langle \dot{x}, p \times B \rangle = -2\kappa,$$

and

$$\begin{aligned}\langle x(t), p \times B \rangle &= \langle x(t_0), p \times B \rangle + \int_{t_0}^t \langle \dot{x}(s), p \times B \rangle ds \\ &= \langle x(t_0), p \times B \rangle - 2\kappa(t) + 2\kappa(t_0).\end{aligned}$$

Thus, if $p \times B \neq 0$, then

$$x(t) = x(t_0) - \frac{2\kappa(t_0)}{|p \times B|^2} p \times B - \left(\frac{1}{|p|^2} \int_{t_0}^t \kappa^2(t) dt \right) p - \frac{2\kappa(t)}{|p \times B|^2} p \times B.$$

- (2) The special case $p \times B = 0$, $p \neq 0$. If $p \times B = 0$, and $p \neq 0$, then p is parallel to B , and equation (81) implies that $\kappa = 0$, so the solution is a straight line.
- (3) If $p = 0$, then equation (81) implies that $\kappa = 0$. If $\kappa = 0$, then \dot{x} is constant, and the motion is again a straight line.

3.4. Closed elastica. As mentioned in the introduction, a significant motivation for this work was understanding how symmetry and conservation laws could be systematically exploited to integrate the elastica equations. In the case of closed elastica, as studied by Langer and Singer [12], it is necessary to add an arclength constraint to the variational problem. This results in studying a modified problem with an undetermined Lagrange multiplier. The modifications to our analysis are straightforward insofar as the use of the conserved quantities is concerned. However, since there is also an immediate integration and reduction of order in the problem, which results in a loss of manifest Euclidean invariance, it seemed preferable to avoid the arclength constrained problem and keep the full symmetry in order to see more clearly how the conservation laws enabled the integration, which is the route taken in the previous section.

In more detail, a slicker, but less transparent approach to the Euler-Lagrange equations runs as follows. For a second order Lagrangian $L(x, \dot{x}, \ddot{x}, t)$ the Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}} \right) = 0,$$

it follows that if

$$\frac{\partial L}{\partial x} \equiv 0,$$

which is the case in the elastica problem,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) \right) = 0,$$

immediately integrates to

$$\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}} \right) = c,$$

where c is a constant.

Now put $q = \dot{x}$, $\dot{q} = \ddot{x}$, and set

$$l(q, \dot{q}, t) := L(x, \dot{x}, \ddot{x}, t) - c \cdot \dot{x}$$

Then the integrated equations are now the Euler-Lagrange equations for the *first order* Lagrangian l

$$\frac{\partial l}{\partial q} - \frac{d}{dt} \left(\frac{\partial l}{\partial \dot{q}} \right) = 0.$$

3.4.1. Reduction for elastica.

Linear momentum. The elastica functional for fixed arclength is

$$\int_{\gamma} \kappa^2 + \lambda \, ds.$$

Here λ is a constant whose value is *a priori* unknown. The curvature κ is

$$\kappa = \frac{|\dot{x} \times \ddot{x}|}{|\dot{x}|^3}$$

so since $ds = |\dot{x}| \, dt$, the reduced Lagrangian is

$$l(q, \dot{q}) = \frac{|q \times \dot{q}|^2}{|q|^5} + \lambda |q| - c \cdot q$$

It remains to compute the Euler-Lagrange equations and look at them in the Frenet frame. The derivatives are

$$\begin{aligned} \frac{\partial l}{\partial q} &= \left(-5 \frac{|q \times \dot{q}|}{|q|^7} + \frac{\lambda}{|q|} \right) q + \frac{2}{|q|^5} \dot{q} \times (q \times \dot{q}) - c, \\ \frac{\partial l}{\partial \dot{q}} &= \frac{2}{|q|^5} q \times (\dot{q} \times q). \end{aligned}$$

Define new variables T, N, B and v by setting $v = |q|$, $T = v^{-1}q$,

$$N = \frac{(q \times \dot{q}) \times q}{|q \times \dot{q}| |q|} = \frac{|q|^2 \dot{q} - \langle q, \dot{q} \rangle q}{|q \times \dot{q}| |q|}, \quad B = T \times N.$$

This implies

$$\dot{q} = \dot{v}T + v^2 \kappa N, \quad \frac{\partial l}{\partial \dot{q}} = \frac{2}{v} \kappa N.$$

If we recall the Frenet equations then it follows that

$$\frac{\partial l}{\partial q} = (-3\kappa^2 + \lambda)T - 2\frac{\kappa \dot{v}}{v^2}N - c,$$

and

$$\frac{d}{dt} \left(\frac{\partial l}{\partial \dot{q}} \right) = 2 \left(-\kappa^2 T + \left(\frac{\kappa}{v} \right) N + \kappa \tau B \right).$$

This implies that the Euler-Lagrange equations are

$$(\lambda - \kappa^2)T - 2\frac{\kappa}{v}N - 2\kappa\tau B = c.$$

Taking the inner product of this equation with itself and choosing the arclength parametrization (so $v = 1$) yields

$$4(\kappa')^2 + (\lambda - \kappa^2)^2 + 4\kappa^2\tau^2 = c^2.$$

Angular momentum. The reduced Lagrangian l is not invariant under the rotation group $\text{SO}(3)$, but it is invariant under the $\text{SO}(2)$ subgroup generated by the vector field

$$X = (c \times q) \frac{\partial}{\partial q}$$

which is rotation about the axis defined by $c \neq 0$. The associated conserved momentum is

$$j = \langle p dq, X \rangle = \langle 2v^{-1} \kappa N, c \times q \rangle.$$

The only nonzero component of this contraction is in the N component of $c \times q$, and since the Frenet frame is orthonormal,

$$j = 2\kappa \langle c, B \rangle.$$

Taking the inner product of c with the Euler-Lagrange equations gives

$$\langle c, B \rangle = -2\kappa\tau,$$

and this implies that the conserved angular momentum j is

$$j = -4\kappa^2\tau.$$

Thus $4\kappa^2\tau^2 = j^2/4\kappa^2$, and substituting back into the equation for κ' yields

$$4(\kappa')^2 + (\lambda - \kappa^2)^2 + \frac{j^2}{4\kappa^2} = c^2.$$

This recovers equations (3) and (4) of FOLTINEK [6], together with the interpretation of c as linear momentum and j as angular momentum.

4. ELASTICA AS A CONSTRAINED HAMILTONIAN SYSTEM

4.1. Range of the Legendre transformation. In this section we discuss the range of the Legendre transformation

$$\mathcal{L} : J_0^3 \rightarrow T^*J_0^1 : (t, x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \mapsto (t, x, \dot{x}, p_t, p_x, p_{\dot{x}})$$

for elastica with Lagrangian

$$L(x, \dot{x}, \ddot{x}) = \frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^3}. \quad (82)$$

Here, the subscript 0 denotes that $\dot{x} \neq 0$, and

$$p_{\dot{x}} = \frac{\partial L}{\partial \ddot{x}} = 2 \frac{\ddot{x}^\perp}{|\dot{x}|^3}, \quad (83)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} = -\frac{2}{|\dot{x}|^3} \ddot{\ddot{x}}^\perp + \frac{6}{|\dot{x}|^5} \langle \dot{x}, \ddot{x} \rangle \ddot{x}^\perp - \frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^5} \dot{x}, \quad (84)$$

$$p_t = L - \langle p_x, \dot{x} \rangle - \langle p_{\dot{x}}, \ddot{x} \rangle = 0. \quad (85)$$

The $\text{Diff}_+ \mathbf{R}$ -invariance of $L dt$ is responsible for the vanishing of p_t above, and it implies that the variable $p_{\dot{x}}$ satisfies the equation

$$\langle p_{\dot{x}}, \dot{x} \rangle = 0 \quad (86)$$

(see remark 2.31.) Note that equations (83) and (84) imply

$$\langle p_{\dot{x}}, p_{\dot{x}} \rangle = 4 \frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^6} = \frac{4}{|\dot{x}|^3} L, \quad (87)$$

$$\langle p_{\dot{x}}, \ddot{x} \rangle = \left\langle 2 \frac{\ddot{x}^\perp}{|\dot{x}|^3}, \ddot{x} \right\rangle = 2 \frac{\langle \ddot{x}^\perp, \ddot{x} \rangle}{|\dot{x}|^3} = 2 \frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^3} = 2L = \frac{1}{2} |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle, \quad (88)$$

$$\langle p_x, \dot{x} \rangle = \left\langle -\frac{2}{|\dot{x}|^3} \ddot{x}^\perp + \frac{6}{|\dot{x}|^5} \langle \dot{x}, \ddot{x} \rangle \ddot{x}^\perp - \frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^5} \dot{x}, \dot{x} \right\rangle = -\frac{1}{4} |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle. \quad (89)$$

Equation (85), written in terms of the variables on $T^*J_0^1$, reads

$$H(x, \dot{x}, p_x, p_{\dot{x}}) = -\frac{1}{4} |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{1}{2} |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle - \frac{1}{4} |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle = 0.$$

Hence, it does not introduce further restrictions of the variables $(t, x, \dot{x}, p_t, p_x, p_{\dot{x}})$. Equations (88) and (89) lead to the new constraint equation

$$\langle p_{\dot{x}}, \ddot{x} \rangle + 2 \langle p_x, \dot{x} \rangle = 0, \quad (90)$$

while equation (90) in terms of the variables on $T^*J_0^1$ is

$$|\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0. \quad (91)$$

THEOREM 4.1. *The range of the Legendre transformation $\mathcal{L} : J_0^3 \rightarrow T^*J_0^1$ is the common zero set of the three functions p_t , $\langle p_{\dot{x}}, \dot{x} \rangle$ and $\langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{4}{|\dot{x}|^3} \langle p_x, \dot{x} \rangle$. That is,*

$$\text{range } \mathcal{L} = \{(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1 \mid p_t = \langle p_{\dot{x}}, \dot{x} \rangle = |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0\}. \quad (92)$$

Proof. Equations (85), (86) and (91) imply that

$$\text{range } \mathcal{L} \subseteq \{(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1 \mid p_t = \langle p_{\dot{x}}, \dot{x} \rangle = |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0\}.$$

Suppose $(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1$ is such that $p_t = 0$, $\langle p_{\dot{x}}, \dot{x} \rangle = 0$ and $|\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0$. Since $\langle p_{\dot{x}}, \dot{x} \rangle = 0$, it follows that $p_{\dot{x}} = p_{\dot{x}}^\perp$, and equation (83) implies $\ddot{x}^\perp = \frac{1}{2} |\dot{x}|^3 p_{\dot{x}}$. By definition, the vanishing of $|\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle$ is equivalent to equation (90); that is $\langle p_{\dot{x}}, \ddot{x} \rangle + 2 \langle p_x, \dot{x} \rangle = 0$. Splitting equation (84) into its components perpendicular and parallel to \dot{x} gives

$$p_x^\perp = -\frac{2}{|\dot{x}|^3} \ddot{x}^\perp + \frac{6}{|\dot{x}|^5} \langle \dot{x}, \ddot{x} \rangle \ddot{x}^\perp, \quad (93)$$

$$p_x^\parallel = -\frac{\langle \ddot{x}^\perp, \ddot{x}^\perp \rangle}{|\dot{x}|^5} \dot{x}. \quad (94)$$

Equation (93) gives

$$\begin{aligned} \frac{2}{|\dot{x}|^3} \ddot{x}^\perp &= -p_x^\perp + \frac{6}{|\dot{x}|^5} \langle \dot{x}, \ddot{x} \rangle \ddot{x}^\perp = -p_x^\perp + \frac{6}{|\dot{x}|^5} \langle \dot{x}, \ddot{x} \rangle \frac{1}{2} |\dot{x}|^3 p_{\dot{x}} \\ &= -p_x^\perp + \frac{3}{|\dot{x}|^2} \langle \dot{x}, \ddot{x} \rangle p_{\dot{x}}, \end{aligned}$$

where $\langle \dot{x}, \ddot{x} \rangle$ is arbitrary. Equation (94) is equivalent to equation (90) because

$$\langle p_x, \dot{x} \rangle = \langle p_x^{\parallel}, \dot{x} \rangle = -\frac{\langle \ddot{x}^{\perp}, \ddot{x}^{\perp} \rangle}{|\dot{x}|^5} \langle \dot{x}, \dot{x} \rangle = -\frac{\langle \ddot{x}^{\perp}, \ddot{x}^{\perp} \rangle}{|\dot{x}|^3} = -\frac{1}{2} \langle p_{\dot{x}}, \ddot{x} \rangle$$

by equation (88).

The above argument shows that the fibre of \mathcal{L} over the point $(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1$ such that $p_t = 0$, $\langle p_{\dot{x}}, \dot{x} \rangle = 0$ and $|\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0$ is not empty. In fact,

$$\mathcal{L}^{-1}(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) = \{(t, x, \dot{x}, \ddot{x}^{\perp} + \ddot{x}^{\parallel}, \ddot{x}^{\perp} + \ddot{x}^{\parallel})\},$$

where

$$\begin{aligned} \ddot{x}^{\perp} &= \frac{1}{2} |\dot{x}|^3 p_{\dot{x}}, \\ \ddot{x}^{\parallel} &= \frac{1}{2} \left(-|\dot{x}|^3 p_x^{\perp} + 3 |\dot{x}| \langle \dot{x}, \ddot{x}^{\parallel} \rangle p_{\dot{x}} \right), \end{aligned}$$

and \ddot{x}^{\parallel} and \ddot{x}^{\perp} are arbitrary. q.e.d.

THEOREM 4.2. *The range of the Legendre transformation is a submanifold of $T^*J_0^1$.*

Proof. Fix $(t, x, \dot{x}) \in J_0^1$. The constraint equations

$$\begin{aligned} \langle p_{\dot{x}}, \dot{x} \rangle &= 0, \\ \langle \dot{x}, \dot{x} \rangle^{3/2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle &= 0, \end{aligned}$$

give

$$\begin{aligned} p_{\dot{x}}^{\parallel} &= 0, \\ p_x^{\parallel} &= -\frac{1}{4} \langle \dot{x}, \dot{x} \rangle^{3/2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle = -\frac{1}{4} \langle \dot{x}, \dot{x} \rangle^{3/2} \langle p_{\dot{x}}^{\perp}, p_{\dot{x}}^{\perp} \rangle. \end{aligned}$$

By assumption, $\dot{x} \neq 0$, which implies that the splitting of vectors into components parallel and perpendicular to \dot{x} is smooth. Therefore,

$$\{(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1 \mid p_t = 0, \langle p_{\dot{x}}, \dot{x} \rangle = 0 \text{ and } |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0\}$$

is equal to

$$\{(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1 \mid p_t = 0, p_{\dot{x}}^{\parallel} = 0 \text{ and } p_x^{\parallel} = -\frac{1}{4} |\dot{x}|^3 \langle p_{\dot{x}}^{\perp}, p_{\dot{x}}^{\perp} \rangle\}$$

and is a submanifold of $T^*J_0^1$. Hence, range \mathcal{L} is a submanifold of $T^*J_0^1$. q.e.d.

Recall that the Liouville form of $T^*J_0^1$ is

$$\theta = p_t dt + p_x dx + p_{\dot{x}} d\dot{x} \tag{95}$$

with exterior derivative

$$\omega = d\theta = dp_t \wedge dt + dp_x \wedge dx + dp_{\dot{x}} \wedge d\dot{x} \tag{96}$$

the canonical symplectic form of $T^*J_0^1$.

For $f \in C^\infty(T^*J_0^1)$, the Hamiltonian vector field of f is the unique vector field X_f on $T^*J_0^1$ such that

$$X_f \lrcorner \omega = -df,$$

where \lrcorner denote the left interior product (contraction). The Poisson bracket of two functions $f_1, f_2 \in C^\infty(T^*J_0^1)$ is given by

$$\{f_1, f_2\} = X_{f_2}(f_1). \quad (97)$$

It is bilinear, antisymmetric, and it satisfies the Jacobi identity

For the sake of future convenience, define the reparametrization-invariant function h by

$$h = \frac{|\dot{x}|^2}{4} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{|\dot{x}|}. \quad (98)$$

Note that h is smooth, because $\dot{x} \neq 0$, and we can use the constraint $h = 0$ instead of $|\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle = 0$ in describing the range of \mathcal{L} . In other words,

$$\text{range } \mathcal{L} = \{(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1 \mid p_t = 0, \langle p_{\dot{x}}, \dot{x} \rangle = 0 \text{ and } h = 0\}. \quad (99)$$

The Hamiltonian vector fields of the constraint functions p_t , $\langle p_{\dot{x}}, \dot{x} \rangle$ and h are

$$\begin{aligned} X_{p_t} &= \frac{\partial}{\partial t}, \\ X_{\langle p_{\dot{x}}, \dot{x} \rangle} &= \dot{x} \frac{\partial}{\partial \dot{x}} - p_{\dot{x}} \frac{\partial}{\partial p_{\dot{x}}}, \\ X_h &= \frac{1}{2} \langle \dot{x}, \dot{x} \rangle p_{\dot{x}} \frac{\partial}{\partial \dot{x}} + \frac{1}{|\dot{x}|} \dot{x} \frac{\partial}{\partial x} - \frac{1}{|\dot{x}|} p_x \frac{\partial}{\partial p_x} + \left(-\frac{1}{2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{|\dot{x}|^3} \right) \dot{x} \frac{\partial}{\partial p_{\dot{x}}} \end{aligned}$$

Note that all the Poisson brackets of the constraint functions vanish identically

$$\{\langle p_{\dot{x}}, \dot{x} \rangle, p_t\} = \{h, p_t\} = \{h, \langle p_{\dot{x}}, \dot{x} \rangle\} = 0. \quad (100)$$

This implies that range \mathcal{L} is a coisotropic submanifold of $(T^*J_0^1, \omega)$.

4.2. Action of $\text{Diff}_+ \mathbf{R}$ on $T^*J_0^1$. Recall that for $X = \tau \partial_t \in \text{diff}_+ \mathbf{R}$, the action of the one-parameter subgroup $\exp sX$ on J_0^1 is generated by the vector field $X^1 = \tau \frac{\partial}{\partial t} - \dot{\tau} \dot{x} \frac{\partial}{\partial \dot{x}}$. The lifted action of $\exp sX$ on $T^*J_0^1$ is generated by the Hamiltonian vector field $X_{\mathcal{J}_\tau}$, where

$$\mathcal{J}_\tau(t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) = \langle p_t dt + p_x dx + p_{\dot{x}} d\dot{x}, X_\tau(t, x, \dot{x}) \rangle = \tau(t) p_t - \dot{\tau}(t) \langle p_{\dot{x}}, \dot{x} \rangle.$$

The map

$$\mathcal{J}_{\text{diff}} : \text{diff}_+ \mathbf{R} \rightarrow T^*J_0^1 : \tau \frac{\partial}{\partial t} \mapsto \mathcal{J}_\tau = \tau(t) p_t - \dot{\tau}(t) \langle p_{\dot{x}}, \dot{x} \rangle$$

may be interpreted as the momentum map for the action of the group $\text{Diff}_+ \mathbf{R}$ on $T^*J_0^1$. Writing it this way avoids unnecessary discussion about the topology of the dual of the Lie algebra $\text{diff}_+ \mathbf{R}$. The constraint equations $p_t = 0$ and $\langle p_{\dot{x}}, \dot{x} \rangle = 0$ imply that $\mathcal{J}_{\text{diff}}$ vanishes on range \mathcal{L} . In other words,

$$\text{range } \mathcal{L} \subseteq \mathcal{J}_{\text{diff}}^{-1}(0).$$

PROPOSITION 4.3. $\mathcal{J}_{\text{diff}}^{-1}(0)$ is a coisotropic submanifold of $T^*J_0^1$. The null distribution of the pullback of ω to $\mathcal{J}_{\text{diff}}^{-1}(0)$ is spanned by the Hamiltonian vector fields X_{p_t} and $X_{\langle p_{\dot{x}}, \dot{x} \rangle}$.

Proof. This follows from the proof of theorem 4.2 and equation (100). q.e.d.

Integral curves of the Hamiltonian vector field $X_{p_t} = \frac{\partial}{\partial t}$ are lines parallel to the t -axis. Integral curves of $X_{\langle p_{\dot{x}}, \dot{x} \rangle}$ satisfy equations

$$\begin{aligned} \frac{d}{ds} \dot{x}(s) &= \dot{x}(s), \\ \frac{d}{ds} p_{\dot{x}}(s) &= -p_{\dot{x}}(s). \end{aligned}$$

Hence, for each $\mathbf{p} = (t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in T^*J_0^1$, the integral manifold of the distribution on $T^*J_0^1$ spanned by X_{p_t} and $X_{\langle p_{\dot{x}}, \dot{x} \rangle}$ that passes through \mathbf{p} is

$$O_{\mathbf{p}} = \{(u, x, e^s \dot{x}, p_t, p_x, e^{-s} p_{\dot{x}}) \mid (u, s) \in \mathbf{R}^2\}. \quad (101)$$

THEOREM 4.4. *For each $\mathbf{p} \in \mathcal{J}_{\text{diff}}^{-1}(0) \subset T^*J_0^1$, the orbit of the Lie algebra $\text{diff}_+ \mathbf{R}$ through \mathbf{p} and of the reparametrization group $\text{Diff}_+ \mathbf{R}$ coincides with the integral manifold $O_{\mathbf{p}}$ given by equation (101), where $p_t = 0$.*

Proof. Orbits of the action of the Lie algebra $\text{diff}_+ \mathbf{R}$ on $T^*J_0^1$ are orbits (accessible sets) of the family $\{X_{\mathcal{J}_{\tau}} \mid \tau \frac{\partial}{\partial t} \in \text{diff}_+ \mathbf{R}\}$ of Hamiltonian vector fields on $T^*J_0^1$. Since $X_{\mathcal{J}_{\tau}} = X_{\tau p_t} - X_{\tau \langle p_{\dot{x}}, \dot{x} \rangle}$ and \mathcal{J}_{τ} vanishes on $\mathcal{J}_{\text{diff}}^{-1}(0)$, it follows that the restriction of $X_{\mathcal{J}_{\tau}}$ to $\mathcal{J}_{\text{diff}}^{-1}(0)$ is

$$X_{\mathcal{J}_{\tau}| \mathcal{J}_{\text{diff}}^{-1}(0)} = \tau X_{p_t| \mathcal{J}_{\text{diff}}^{-1}(0)} - \tau X_{\langle p_{\dot{x}}, \dot{x} \rangle| \mathcal{J}_{\text{diff}}^{-1}(0)}.$$

Therefore, $X_{\mathcal{J}_{\tau}| \mathcal{J}_{\text{diff}}^{-1}(0)}$ is contained in the distribution spanned by $X_{p_t| \mathcal{J}_{\text{diff}}^{-1}(0)}$ and $X_{\langle p_{\dot{x}}, \dot{x} \rangle| \mathcal{J}_{\text{diff}}^{-1}(0)}$. Hence, for each $\mathbf{p} \in \mathcal{J}_{\text{diff}}^{-1}(0)$, the orbit of $\text{diff}_+ \mathbf{R}$ through \mathbf{p} coincides with the integral manifold $O_{\mathbf{p}}$ given by equation (101).

The reparametrization group $\text{Diff}_+ \mathbf{R}$ acts on J_0^1 by

$$\text{Diff}_+ \mathbf{R} \times J_0^1 \rightarrow J_0^1 : (\varphi, (t, x, \dot{x})) \mapsto \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)} \right),$$

where φ is a smooth function on \mathbf{R} such that $\dot{\varphi}(t) > 0$ for $t \in \mathbf{R}$. The lift of this action to $T^*J_0^1$ is

$$\text{Diff}_+ \mathbf{R} \times T^*J_0^1 \rightarrow T^*J_0^1 : (\varphi, \mathbf{p}) \mapsto \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)}, \frac{p_t}{\dot{\varphi}(t)} + \frac{\langle p_{\dot{x}}, \dot{x} \rangle \ddot{\varphi}(t)}{\dot{\varphi}(t)^2}, p_x, \dot{\varphi}(t) p_{\dot{x}} \right).$$

It preserves $\mathcal{J}_{\text{diff}}^{-1}(0)$. Hence, the orbit of $\text{Diff}_+ \mathbf{R}$ is

$$\{(\varphi(t), x, \dot{\varphi}^{-1}(t) \dot{x}, \dot{\varphi}(t)^{-1} p_t, p_x, \dot{\varphi}(t) p_{\dot{x}}) \in T^*J_0^1 \mid \varphi \in C^\infty(\mathbf{R}), \dot{\varphi}(t) > 0\}.$$

The action of $\text{Diff}_+ \mathbf{R}$ preserves $\mathcal{J}_{\text{diff}}^{-1}(0)$, given by $p_t = 0$ and $\langle p_{\dot{x}}, \dot{x} \rangle = 0$. Hence, orbits of $\text{Diff}_+ \mathbf{R}$ contained in $\mathcal{J}_{\text{diff}}^{-1}(0)$ are

$$\{(\varphi(t), x, \dot{\varphi}^{-1}(t) \dot{x}, 0, p_x, \dot{\varphi}(t) p_{\dot{x}}) \in T^*J_0^1 \mid \langle p_{\dot{x}}, \dot{x} \rangle = 0, \varphi \in C^\infty(\mathbf{R}), \dot{\varphi}(t) > 0\}.$$

For each t , $\varphi(t) = u$ and $\dot{\varphi}(t) = -s$, are independent. Therefore, orbits of $\text{Diff}_+ \mathbf{R}$ contained in $\mathcal{J}_{\text{diff}}^{-1}(0)$ coincide with the corresponding integral manifolds given by equation (101). q.e.d.

4.3. Reduction of $\text{Diff}_+ \mathbf{R}$ symmetries. In this section, we discuss the space

$$R = \mathcal{J}_{\text{diff}}^{-1}(0) / \text{Diff}_+ \mathbf{R}$$

of $\text{Diff}_+ \mathbf{R}$ -orbits in $\mathcal{J}_{\text{diff}}^{-1}(0)$. According to Theorem 4.4, the reduced phase space R is the space of integral manifolds in $\mathcal{J}_{\text{diff}}^{-1}(0)$ of the distribution spanned by X_{p_t} and $X_{\langle p_{\dot{x}}, \dot{x} \rangle}$. We have shown that the orbit of the vector fields $\{X_{p_t}, X_{\langle p_{\dot{x}}, \dot{x} \rangle}\}$ through $\mathbf{p} = (t, x, \dot{x}, p_t, p_x, p_{\dot{x}}) \in \mathcal{J}_{\text{diff}}^{-1}(0)$ is

$$O_{\mathbf{p}} = \{(u, x, e^s \dot{x}, p_t, p_x, e^{-s} p_{\dot{x}}) \mid (u, s) \in \mathbf{R}^2\}. \quad (102)$$

We are going to show R is a quotient manifold of $\mathcal{J}_{\text{diff}}^{-1}(0)$, which will imply that R has a unique symplectic form ω_R such that

$$\rho^* \omega_R = \iota^* \omega, \quad (103)$$

where $\iota : \mathcal{J}_{\text{diff}}^{-1}(0) \rightarrow T^*J_0^1$ is the inclusion map.

In order to parametrize the reduced phase space R , define spherical coordinates $(\dot{r}, \dot{\alpha}, \dot{\beta})$ by

$$\begin{aligned} \dot{x}^1 &= \dot{r} \sin \dot{\beta} \cos \dot{\alpha}, \\ \dot{x}^2 &= \dot{r} \sin \dot{\beta} \sin \dot{\alpha}, \\ \dot{x}^3 &= \dot{r} \cos \dot{\beta}, \end{aligned} \quad (104)$$

together with the dual momentum variables $(p_{\dot{r}}, p_{\dot{\beta}}, p_{\dot{\alpha}})$ defined by

$$p_{\dot{x}} d\dot{x} = p_{\dot{r}} d\dot{r} + p_{\dot{\beta}} d\dot{\beta} + p_{\dot{\alpha}} d\dot{\alpha}. \quad (105)$$

PROPOSITION 4.5. $p_{\dot{r}} = \langle p_{\dot{x}}, \dot{x} \rangle / \dot{r}$.

Proof. This is a simple verification. q.e.d.

Denote by $\rho : \mathcal{J}_{\text{diff}}^{-1}(0) \rightarrow R$ the reduction map associating to each point in $\mathcal{J}_{\text{diff}}^{-1}(0)$ the orbit of $\{X_{p_t}, X_{\langle p_{\dot{x}}, \dot{x} \rangle}\}$ through that point. Let

$$S = \{(x, \dot{x}) \in T\mathbf{R}^3 \mid |\dot{x}| = 1\}$$

be the unit sphere bundle over \mathbf{R}^3 parametrized by coordinates $(x, \dot{\alpha}, \dot{\beta})$. The Liouville form of T^*S is

$$\theta_S = p_x dx + p_{\dot{\beta}} d\dot{\beta} + p_{\dot{\alpha}} d\dot{\alpha}, \quad (106)$$

and

$$\omega_S = d\theta_S$$

is the canonical symplectic form of T^*S .

PROPOSITION 4.6. *There is a unique symplectomorphism $\kappa : (R, \omega_R) \rightarrow (T^*S, \omega_S)$ such that*

$$\kappa \circ \rho : \mathcal{J}_{\text{diff}}^{-1}(0) \rightarrow T^*S : (t, x, \dot{x}, 0, p_x, p_{\dot{x}}) \mapsto (x, \dot{\beta}, \dot{\alpha}, p_x, p_{\dot{\beta}}, p_{\dot{\alpha}}),$$

where the $(\dot{\beta}, \dot{\alpha}, p_{\dot{\beta}}, p_{\dot{\alpha}})$ are related to $(x, \dot{x}, p_x, p_{\dot{x}})$ by equations (104) and (105).

Proof. Consider first the space $R_1 = p_t^{-1}(0)/X_{p_t}$ of integral curves of X_{p_t} in $p_t^{-1}(0)$. It is a quotient manifold of $p_t^{-1}(0)$ with projection map

$$\rho_1 : p_t^{-1}(0) \rightarrow R_1 : (t, x, \dot{x}, 0, p_x, p_{\dot{x}}) \mapsto (x, \dot{x}, p_x, p_{\dot{x}}).$$

Moreover, it is a symplectic manifold with the symplectic form

$$\omega_1 = dp_x \wedge dx + dp_{\dot{x}} \wedge d\dot{x}.$$

The constraint function $\langle p_{\dot{x}}, \dot{x} \rangle$ is left invariant by the action X_{p_t} , and pushes forward to a function on R_1 , denoted by $\langle p_{\dot{x}}, \dot{x} \rangle_1$. That is, $\langle p_{\dot{x}}, \dot{x} \rangle = \rho_1^* \langle p_{\dot{x}}, \dot{x} \rangle_1$. Moreover, the Hamiltonian vector field $X_{\langle p_{\dot{x}}, \dot{x} \rangle}$ restricted to $p_t^{-1}(0)$ pushes forward to the Hamiltonian vector field on R_1 corresponding to the function $\langle p_{\dot{x}}, \dot{x} \rangle_1$ on R_1 . Denote this vector field by $X_{\langle p_{\dot{x}}, \dot{x} \rangle_1}$.

By definition, $\dot{r} = |\dot{x}| \neq 0$ on $T^*J_0^1$. Since $p_{\dot{r}}\dot{r} = \langle p_{\dot{x}}, \dot{x} \rangle$, it follows that on $\rho_1(\mathcal{J}_{\text{diff}}^{-1}(0))$ the Hamiltonian vector field of $\langle p_{\dot{x}}, \dot{x} \rangle_1$ is proportional to the Hamiltonian vector field $X_{p_r} = \frac{\partial}{\partial \dot{r}}$. Therefore, the space $R_2 = \langle p_{\dot{x}}, \dot{x} \rangle_1^{-1}(0)/X_{\langle p_{\dot{x}}, \dot{x} \rangle_1}$ of orbits of $X_{\langle p_{\dot{x}}, \dot{x} \rangle_1}$ in $\langle p_{\dot{x}}, \dot{x} \rangle_1^{-1}(0)$ can be parametrized by $(x, \dot{\beta}, \dot{\alpha}, p_x, p_{\dot{\beta}}, p_{\dot{\alpha}})$. It is a symplectic manifold with the symplectic form

$$\omega_2 = dp_x \wedge dx + dp_{\dot{\beta}} \wedge d\dot{\beta} + dp_{\dot{\alpha}} \wedge d\dot{\alpha}.$$

The coordinates $(x, \dot{\beta}, \dot{\alpha}, p_x, p_{\dot{\beta}}, p_{\dot{\alpha}})$ define a symplectomorphism between (R_2, ω_2) and (T^*S, ω_S) , where ω_S is the pullback to T^*S of the canonical symplectic form on $T^*(T\mathbf{R}^3)$. However, $R = \mathcal{J}_{\text{diff}}^{-1}(0)/\{X_{p_t}, X_{\langle p_{\dot{x}}, \dot{x} \rangle}\}$ with the symplectic form ω_R is naturally symplectomorphic to (R_2, ω_2) . Hence, (R, ω_R) is symplectomorphic to (T^*S, ω_S) . q.e.d.

It follows from Proposition 4.6 that we may identify (R, ω_R) with (T^*S, ω_S) .

The action of the Euclidean group $\text{SE}(3)$ on \mathbf{R}^3 induces a Hamiltonian action of E on $T^*J_0^1$ generated by the Hamiltonian vector fields X_{p_x} , $X_{p_{\dot{\beta}}}$ and $X_{p_{\dot{\alpha}}}$. This action preserves the constraint functions p_t , $\langle p_{\dot{x}}, \dot{x} \rangle$ and h . In particular, it induces an action of $\text{SE}(3)$ on the zero level set $\mathcal{J}_{\text{diff}}^{-1}(0)$ of the momentum map for the action of $\text{diff}_+ \mathbf{R}$. On the other hand, the action of E on \mathbf{R}^3 induces a Hamiltonian action of $\text{SE}(3)$ on T^*S , presented as

$$T^*S = \{(x, \dot{x}, p_x, p_{\dot{x}}) \in T^*(T\mathbf{R}^3) \mid |\dot{x}| = 1, p_{\dot{r}} = 0\},$$

which is generated by the Hamiltonian vector fields of p_x , $p_{\dot{\beta}}$, and $p_{\dot{\alpha}}$ considered as functions on T^*S . Moreover, these actions of $\text{SE}(3)$ are intertwined by the reduction map $\rho : \mathcal{J}_{\text{diff}}^{-1}(0) \rightarrow R$ followed by the identification $R \cong T^*S$.

4.4. Hamiltonian dynamics. The range of the Legendre transformation is characterized as

$$\text{range } \mathcal{L} = \mathcal{J}_{\text{diff}}^{-1}(0) \cap h^{-1}(0),$$

where

$$h = \frac{|\dot{x}|^2}{4} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{|\dot{x}|}.$$

The Hamiltonian vector field of h is

$$X_h = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle p_{\dot{x}} \frac{\partial}{\partial \dot{x}} + \frac{1}{\langle \dot{x}, \dot{x} \rangle^{1/2}} \dot{x} \frac{\partial}{\partial x} - \frac{1}{\langle \dot{x}, \dot{x} \rangle^{1/2}} p_x \frac{\partial}{\partial p_{\dot{x}}} + \left(-\frac{1}{2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{\langle \dot{x}, \dot{x} \rangle^{3/2}} \right) \dot{x} \frac{\partial}{\partial p_x}.$$

In order to find the integral curves of X_h on the range of \mathcal{L} , observe that they satisfy the equations

$$\begin{aligned} \frac{d}{ds} x &= \frac{1}{\langle \dot{x}, \dot{x} \rangle^{1/2}} \dot{x}, \\ \frac{d}{ds} \dot{x} &= \frac{1}{2} \langle \dot{x}, \dot{x} \rangle p_{\dot{x}}, \\ \frac{d}{ds} p_{\dot{x}} &= -\frac{1}{\langle \dot{x}, \dot{x} \rangle^{1/2}} p_x + \left(-\frac{1}{2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{\langle \dot{x}, \dot{x} \rangle^{3/2}} \right) \dot{x}, \\ \frac{d}{ds} p_x &= 0, \\ \frac{d}{ds} t &= 0. \end{aligned}$$

Multiplying equation $\frac{d}{ds} \dot{x} = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle p_{\dot{x}}$ by \dot{x} , and using the constraint equation $\langle p_{\dot{x}}, \dot{x} \rangle = 0$, yields $\frac{d}{ds} \langle \dot{x}, \dot{x} \rangle = 0$, hence $\langle \dot{x}(s), \dot{x}(s) \rangle = |\dot{x}_0|^2$. Since h is reparametrization invariant, without loss of generality, we may assume that $|\dot{x}_0| = 1$, that is, the arclength parametrization. Moreover, on the range of \mathcal{L} , $h = 0$, which implies that $\frac{1}{2} \langle p_{\dot{x}}, p_{\dot{x}} \rangle = -2 \langle p_x, \dot{x} \rangle$. This leads to

$$\frac{d}{ds} x = \dot{x}, \tag{107}$$

$$\frac{d}{ds} \dot{x} = \frac{1}{2} p_{\dot{x}}, \tag{108}$$

$$\frac{d}{ds} p_{\dot{x}} = -p_x + 3 \langle p_x, \dot{x} \rangle \dot{x}, \tag{109}$$

$$\frac{d}{ds} p_x = 0, \tag{110}$$

$$\frac{d}{ds} t = 0. \tag{111}$$

Equation (110) implies that p_x is constant. The angular momentum is given by $l = x \times p_x + \dot{x} \times p_{\dot{x}}$ (see equation (61.)) Hence,

$$\begin{aligned} \frac{d}{ds} l &= \left(\frac{d}{ds} x \right) \times p_x + x \times \left(\frac{d}{ds} p_x \right) + \left(\frac{d}{ds} \dot{x} \right) \times p_{\dot{x}} + \dot{x} \times \left(\frac{d}{ds} p_{\dot{x}} \right) \\ &= \dot{x} \times p_x + \frac{1}{2} p_{\dot{x}} \times p_{\dot{x}} + \dot{x} \times (-p_x + 3 \langle p_x, \dot{x} \rangle \dot{x}) \\ &= 0, \end{aligned}$$

and so is conserved. Therefore, $\langle p_x, l \rangle = \langle p_x, \dot{x} \times p_{\dot{x}} \rangle$ is also conserved. Multiplying equations (108) and (109) by p_x yields

$$\begin{aligned} \frac{d}{ds} \langle p_x, \dot{x} \rangle &= \langle p_x, p_{\dot{x}} \rangle, \\ \frac{d}{ds} \langle p_x, p_{\dot{x}} \rangle &= -\langle p_x, p_x \rangle + 3 \langle p_x, \dot{x} \rangle^2, \end{aligned}$$

or

$$\frac{d^2}{ds^2} \langle p_x, \dot{x} \rangle = -\langle p_x, p_x \rangle + 3 \langle p_x, \dot{x} \rangle^2.$$

Multiplying by $\frac{d}{ds} \langle p_x, \dot{x} \rangle$ and integrating gives

$$\frac{1}{2} \left(\frac{d \langle p_x, \dot{x} \rangle}{ds} \right)^2 = -\langle p_x, p_x \rangle \langle p_x, \dot{x} \rangle + \langle p_x, \dot{x} \rangle^3 + \text{constant},$$

which can be integrated since it is separable. If $p_x \neq 0$, then this equation gives the component

$$\dot{x}^{\parallel} = \frac{\langle p_x, \dot{x} \rangle}{|p_x|^2} p_x$$

of \dot{x} parallel to p_x . Integrating $\dot{x}^{\parallel}(s)$ yields the component of the motion in the direction of p_x . Returning to equations (108) and (109) gives

$$\frac{d}{ds} \dot{x} = \frac{1}{2} p_{\dot{x}}, \quad (112)$$

$$\frac{d}{ds} p_{\dot{x}} = -p_x + 3 \langle p_x, \dot{x} \rangle \dot{x}, \quad (113)$$

where $\langle p_x, \dot{x} \rangle$ is assumed known from the discussion above. Hence,

$$\frac{d^2}{ds^2} \dot{x} = -\frac{1}{2} p_x + \frac{3}{2} \langle p_x, \dot{x} \rangle \dot{x}. \quad (114)$$

Writing \dot{x} and p_x in terms of their components \dot{x}^i and p_{x^i} ,

$$\frac{d^2}{ds^2} \dot{x}^i = -\frac{1}{2} p_{x^i} + \frac{3}{2} \langle p_x, \dot{x} \rangle \dot{x}^i. \quad (115)$$

Division by \dot{x}_i implies

$$\frac{d}{ds} \left(\ln \left| \frac{d}{ds} \dot{x}^i \right| \right) = \frac{1}{\frac{d\dot{x}^i}{ds}} \frac{d^2 \dot{x}^i}{ds^2} = -\frac{p_{x^i}}{2 \frac{d\dot{x}^i}{ds}} + \frac{3}{2} \langle p_x, \dot{x} \rangle,$$

which implies

$$\ln \left| \frac{d}{ds} \dot{x}^i \right| = -\frac{p_{x^i}}{2} \ln |\dot{x}^i| + \frac{3}{2} \int \langle p_x, \dot{x} \rangle(s) ds$$

or

$$\ln \left| \dot{x}^{p_{x^i}/2} \frac{d}{ds} \dot{x}^i \right| = \frac{3}{2} \int \langle p_x, \dot{x} \rangle(s) ds,$$

so that

$$(\dot{x}^i)^{p_{x^i}/2} \frac{d}{ds} \dot{x}^i = c \exp \left(\int f(s) ds \right),$$

where c is a constant dependent on the initial data. Integrating once more yields

$$\frac{1}{c+1}(\dot{x}^i)^{1+p_{xi}/2} = c \int \exp\left(\int f(s) ds\right) ds$$

if $p_{xi}/2 \neq -1$, and

$$\ln|\dot{x}^i| = \int \exp\left(\int f(s) ds\right) ds$$

if $p_{xi}/2 = -1$.

REMARK 4.7. The Hamiltonian vector field of $p_t + h$ is

$$X_{p_t+h} = X_{p_t} + X_h = \frac{\partial}{\partial t} + \frac{1}{2}p_{\dot{x}}\frac{\partial}{\partial \dot{x}} + \frac{1}{|\dot{x}|^3}\dot{x}\frac{\partial}{\partial x} - \frac{1}{|\dot{x}|^3}p_x\frac{\partial}{\partial p_{\dot{x}}} + 3\frac{\langle p_x, \dot{x} \rangle}{|\dot{x}|^5}\dot{x}\frac{\partial}{\partial p_{\dot{x}}}.$$

In the arclength parametrization, it is

$$\begin{aligned} \frac{d}{ds}t &= 1, \\ \frac{d}{ds}x &= \dot{x}, \\ \frac{d}{ds}\dot{x} &= \frac{1}{2}p_{\dot{x}}, \\ \frac{d}{ds}p_{\dot{x}} &= -p_x + 3\langle p_x, \dot{x} \rangle \dot{x}, \\ \frac{d}{ds}p_x &= 0. \end{aligned}$$

Hence, $t = t_0 + s$, and the solutions of this system can be obtained from the solutions for integral curves of X_h by replacing s by $t - t_0$.

THEOREM 4.8. *If $(t_0, x_0, \dot{x}_0, 0, p_{x_0}, p_{\dot{x}_0}) = \mathcal{L}(t_0, x_0, \dot{x}_0, \ddot{x}_0, \ddot{\ddot{x}}_0)$, the solution of the Euler-Lagrange equation with initial data $(t_0, x_0, \dot{x}_0, \ddot{x}_0, \ddot{\ddot{x}}_0)$ is equivalent to the integral curve of X_{p_t+h} through $(t_0, x_0, \dot{x}_0, 0, p_{x_0}, p_{\dot{x}_0})$ given above.*

Proof. The Euler-Lagrange equations for a second order Lagrangian are

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0.$$

Since $p_{\dot{x}} = \frac{\partial L}{\partial \ddot{x}}$ and $p_x = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}p_{\dot{x}}$, they are $\frac{\partial L}{\partial x} - \frac{d}{dt}p_x = 0$. For elastica,

$$L(x, \dot{x}, \ddot{x}) = \frac{|\ddot{x}|^2}{|\dot{x}|^3} - \frac{\langle \dot{x}, \ddot{x} \rangle^2}{|\dot{x}|^5},$$

$\frac{\partial L}{\partial x} = 0$, and the Euler-Lagrange equations reduce to

$$\frac{d}{dt}p_x = 0.$$

Since $p_x = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}p_{\dot{x}}$, the evolution of $p_{\dot{x}}$ is given by

$$\frac{d}{dt}p_{\dot{x}} = -p_x + \frac{\partial L}{\partial \ddot{x}},$$

where

$$p_{\dot{x}} = \frac{\partial L}{\partial \ddot{x}} = 2 \frac{\ddot{x}}{|\dot{x}|^3} - 2 \frac{\langle \dot{x}, \ddot{x} \rangle \dot{x}}{|\dot{x}|^5},$$

and

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left(\frac{|\ddot{x}|^2}{|\dot{x}|^3} - \frac{\langle \dot{x}, \ddot{x} \rangle^2}{|\dot{x}|^5} \right) = -3 \frac{|\ddot{x}|^2}{|\dot{x}|^5} \dot{x} - 2 \frac{\langle \dot{x}, \ddot{x} \rangle}{|\dot{x}|^5} \ddot{x} + 5 \frac{\langle \dot{x}, \ddot{x} \rangle^2}{|\dot{x}|^7} \dot{x}.$$

In the arclength parametrization $|\dot{x}| = 1$, $\langle \dot{x}, \ddot{x} \rangle = 0$, $p_{\dot{x}} = 2\ddot{x}$ and

$$\frac{\partial L}{\partial \dot{x}} = -3 |\ddot{x}|^2 \dot{x} = -\frac{3}{4} |p_{\dot{x}}|^2 \dot{x} = 3 \langle p_x, \dot{x} \rangle \dot{x}$$

because

$$h = \frac{1}{4} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \langle p_x, \dot{x} \rangle = 0$$

on the range of \mathcal{L} . Thus, the Euler-Lagrange equations of elastica in the arclength parametrization are equivalent to

$$\begin{aligned} \frac{d}{dt} p_x &= 0, \\ \frac{d}{dt} p_{\dot{x}} &= -p_x + 3 \langle p_x, \dot{x} \rangle \dot{x}, \\ \frac{d}{dt} \dot{x} &= \frac{1}{2} p_{\dot{x}}, \\ \frac{d}{dt} x &= \dot{x}. \end{aligned}$$

This system of equations, together with the substitution $t = t_0 + s$, leads to the equation for integral curves of the Hamiltonian vector field of X_{p_t+h} given in the remark above. q.e.d.

5. QUANTIZATION

There are two ways of quantization of a system with constraints: the Bleuler-Gupta quantization of the extended phase space followed by the quantum reduction [1], [8], and Dirac's classical reduction followed by quantization of the reduced phase space studied by Dirac in [4]. Here, we follow the Bleuler-Gupta approach.

The extended phase space $T^*J_0^1$ is the cotangent bundle of an open subset of \mathbf{R}^{14} . Therefore, it is convenient to use the geometric quantization in terms of the vertical polarization tangent to fibres of the cotangent bundle projection. This approach leads to Schrödinger quantization. Since the functions we want to quantize are at most quadratic in momenta, we can use results on Schrödinger quantization as they are presented in texts on quantum mechanics. For technical details see [19].

Consider the space $C_0^\infty(J_0^1) \otimes \mathbf{C}$ of complex-valued compactly supported smooth functions Ψ on J_0^1 endowed with the scalar product

$$(\Psi_1 | \Psi_2) = \int_{J_0^1} \overline{\Psi_1}(t, x, \dot{x}) \Psi_2(t, x, \dot{x}) dt d^3 x d^3 \dot{x}. \quad (116)$$

The completion of $C_0^\infty(J_0^1) \otimes \mathbf{C}$ with respect to the norm given by this scalar product gives rise to the Hilbert space \mathfrak{H}_0 of quantum states of the system.

If $f \in C^\infty(J_0^1)$, then the operator \mathcal{Q}_{π^*f} corresponding to the pullback π^*f of f by the cotangent bundle projection $\pi : T^*J_0^1 \rightarrow J_0^1$ acts on $C^\infty(J_0^1) \otimes \mathbb{C}$ via multiplication by f

$$\mathcal{Q}_{\pi^*f}\Psi = f\Psi. \quad (117)$$

Operators corresponding to canonical momenta act by derivations with respect to the corresponding variables in J_0^1 . In particular, the linear momentum p_x gives rise to

$$\mathcal{Q}_{p_x} = -i\hbar \frac{\partial}{\partial x}, \quad (118)$$

and the angular momentum $l = x \times p_x + \dot{x} \times p_{\dot{x}}$ leads to

$$\mathcal{Q}_l = -i\hbar x \times \frac{\partial}{\partial x} - i\hbar \dot{x} \times \frac{\partial}{\partial \dot{x}}, \quad (119)$$

where \hbar is Planck's constant. Furthermore, $\mathcal{J}_\tau = \tau(t)p_t - \dot{\tau}(t)\langle p_{\dot{x}}, \dot{x} \rangle$ quantizes to

$$\mathcal{Q}_{\mathcal{J}_\tau} = -i\hbar\tau \frac{\partial}{\partial t} + i\hbar\dot{\tau} \left\langle \dot{x}, \frac{\partial}{\partial \dot{x}} \right\rangle - i\hbar\dot{\tau}. \quad (120)$$

Moreover, for $h = \frac{|\dot{x}|^2}{4} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_{\dot{x}}, \dot{x} \rangle}{|\dot{x}|}$, the metric on \mathbf{R}^3 giving the quadratic term has scalar curvature 2, and the corresponding operator is

$$\mathcal{Q}_h = -\frac{\hbar^2}{4} \left(|\dot{x}|^2 \Delta - \frac{1}{3} \right) - \frac{i\hbar}{|\dot{x}|} \left\langle \dot{x}, \frac{\partial}{\partial \dot{x}} \right\rangle, \quad (121)$$

where

$$\Delta = \left\langle \frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial \dot{x}} \right\rangle$$

is the Laplace operator in the variables \dot{x} .⁴ These differential operators on $C^\infty(J_0^1) \otimes \mathbb{C}$ extend to self-adjoint operators on \mathfrak{H}_0 .

5.1. Quantization representation of SE(3). The skew-adjoint operators $\frac{-i}{\hbar}\mathcal{Q}_{p_x}$ and $\frac{-i}{\hbar}\mathcal{Q}_l$ generate a unitary representation of the Euclidean group SE(3) on \mathfrak{H}_0 such that for $g \in \text{SE}(3)$ and $\Psi \in C^\infty(J_0^1) \otimes \mathbb{C}$

$$U_g\Psi(t, x, \dot{x}) = \Phi_{g^{-1}}^*\Psi(t, x, \dot{x}),$$

where

$$\Phi : \text{SE}(3) \times J_0^1 \rightarrow J_0^1$$

is the action of SE(3) on J_0^1 . The invariant vectors of this representation are eigenvectors of $\frac{-i}{\hbar}\mathcal{Q}_{p_x}$ and $\frac{-i}{\hbar}\mathcal{Q}_l$ corresponding to the eigenvalue 0. In other words, invariant vectors Ψ of U are characterized by the equations

$$\begin{aligned} \mathcal{Q}_{p_x}\Psi &= 0, \\ \mathcal{Q}_l\Psi &= 0. \end{aligned}$$

Since SE(3) is not compact, invariant vectors of U are distributions on J_0^1 .

⁴ Correction terms in equations (120) and (121) are consequence of using half-forms in order to ensure that the space \mathfrak{H}_0 of quantum states can be described as the space of square-integrable complex functions on J_0^1 (see [19].)

5.2. Quantization representation of $\text{Diff}_+ \mathbf{R}$. The reparametrization group $\text{Diff}_+ \mathbf{R}$ acts on $C^\infty(J_0^1) \otimes \mathbf{C}$ by the pullback of its action on J_0^1

$$\text{Diff}_+ \mathbf{R} \times (C_0^\infty(J_0^1) \otimes \mathbf{C}) \rightarrow C^\infty(J_0^1) \otimes \mathbf{C} : (\varphi, \Psi) \mapsto (\varphi^{-1})^{1*} \Psi,$$

where φ^{-1} is the inverse of φ , and

$$(\varphi^{-1})^{1*} \Psi(t, x, \dot{x}) = \Psi(\varphi^{-1}(t, x, \dot{x})) = \Psi\left(\varphi^{-1}(t), x, \frac{\dot{x}}{(\varphi^{-1})'(t)}\right).$$

For an infinitesimal diffeomorphism $\varphi_\epsilon(t) = t + \epsilon\tau(t) + \dots$ generated by $\tau(t)\partial_t$,

$$(\varphi_\epsilon^{-1})^{1*} \Psi(t, x, \dot{x}) = \Psi(t, x, \dot{x}) - \epsilon \left(\tau \frac{\partial}{\partial t} - \dot{x} \tau \frac{\partial f}{\partial \dot{x}} \right) \Psi + \dots$$

and

$$\begin{aligned} \frac{d}{d\epsilon} (\varphi_\epsilon^{-1})^{1*} \Psi(t, x, \dot{x})|_{\epsilon=0} &= \left(\tau \frac{\partial}{\partial t} - \dot{x} \tau \frac{\partial f}{\partial \dot{x}} \right) \Psi(t, x, \dot{x}) \\ &= \left(\frac{-i}{\hbar} \mathbf{Q}_{\mathcal{J}_\tau} \right) (t, x, \dot{x}) - \dot{\tau} \Psi(t, x, \dot{x}). \end{aligned} \quad (122)$$

Therefore,

$$\begin{aligned} \left(\frac{-i}{\hbar} \mathbf{Q}_{\mathcal{J}_\tau} \right) (t, x, \dot{x}) &= \frac{d}{d\epsilon} (\varphi_\epsilon^{-1})^{1*} \Psi(t, x, \dot{x})|_{\epsilon=0} + \dot{\tau} \Psi(t, x, \dot{x}) \\ &= \frac{d}{d\epsilon} \{ (\varphi_\epsilon^{-1})^{1*} \Psi(t, x, \dot{x}) + \dot{\varphi}_\epsilon(t) \Psi(t, x, \dot{x}) \}|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \{ (\varphi_\epsilon^{-1})^{1*} [\dot{\varphi}_\epsilon(t) \Psi(t, x, \dot{x})] \}|_{\epsilon=0}. \end{aligned}$$

This establishes

PROPOSITION 5.1. *The operator*

$$\frac{-i}{\hbar} \mathbf{Q}_{\mathcal{J}_\tau} = \tau \frac{\partial}{\partial t} - i\hbar \dot{\tau} \left\langle \dot{x}, \frac{\partial}{\partial \dot{x}} \right\rangle + \dot{\tau}$$

generates the action on $C^\infty(J_0^1) \otimes \mathbf{C}$ of the one-parameter group $\varphi_\epsilon = \exp \epsilon \tau$ given by

$$\Xi : (\varphi_\epsilon, \Psi) \mapsto \Xi_{\varphi_\epsilon} \Psi,$$

where

$$\Xi_{\varphi_\epsilon} \Psi(t, x, \dot{x}) = (\varphi_\epsilon^{-1})^{1*} [\dot{\varphi}_\epsilon \Psi(t, x, \dot{x})].$$

THEOREM 5.2. *The map*

$$\Xi : \text{Diff}_+ \mathbf{R} \times (C^\infty(J_0^1) \otimes \mathbf{C}) \rightarrow C^\infty(J_0^1) \otimes \mathbf{C} : (\varphi, \Psi) \mapsto \Xi_\varphi \Psi,$$

where

$$\Xi_\varphi \Psi(t, x, \dot{x}) = (\varphi^{-1})^{1*} [\dot{\varphi}(t) \Psi(t, x, \dot{x})], \quad (123)$$

is a linear representation of $\text{Diff}_+ \mathbf{R}$ on $C^\infty(J_0^1) \otimes \mathbf{C}$ preserving the scalar product given by equation (116).

Proof. Clearly, Ξ_φ acts linearly on $C^\infty(J_0^1) \otimes \mathbf{C}$. For $\varphi_1, \varphi_2 \in \text{Diff}_+ \mathbf{R}$ and $\Psi \in C^\infty(J_0^1) \otimes \mathbf{C}$,

$$\begin{aligned}
 (\Xi_{\varphi_2} \Xi_{\varphi_1} \Psi) &= \Xi_{\varphi_2} \{(\varphi_1^{-1})^{1*} [\dot{\varphi}_1 \Psi]\} \\
 &= (\varphi_2^{-1})^{1*} \{\dot{\varphi}_2(t) (\varphi_1^{-1})^{1*} [\dot{\varphi}_1 \Psi]\} \\
 &= (\varphi_2^{-1})^* \dot{\varphi}_2 (\varphi_2^{-1})^{1*} \{(\varphi_1^{-1})^{1*} \dot{\varphi}_1 [(\varphi_1^{-1})^{1*} \Psi]\} \\
 &= (\varphi_2^{-1})^* \dot{\varphi}_2 (\varphi_2^{-1})^* (\varphi_1^{-1})^* \dot{\varphi}_1 [(\varphi_2^{-1})^{1*} (\varphi_1^{-1})^{1*} \Psi] \\
 &= (\varphi_2^{-1})^* [\dot{\varphi}_2 (\varphi_1^{-1})^* \dot{\varphi}_1] [((\varphi_2 \circ \varphi_1)^{-1})^* \Psi]
 \end{aligned}$$

But

$$\begin{aligned}
 [(\varphi_2 \circ \varphi_1)^{-1}]^* (\varphi_2 \circ \varphi_1) \dot{\varphi}_1(t) &= (\varphi_2 \circ \varphi_1) ((\varphi_2 \circ \varphi_1)^{-1} \dot{\varphi}_1(t)) \\
 &= (\varphi_2 \circ \varphi_1) \dot{\varphi}_1((\varphi_1^{-1} \circ \varphi_2^{-1})(t)) \\
 &= (\varphi_2 \circ \varphi_1) \dot{\varphi}_1(\varphi_1^{-1}(\varphi_2^{-1}(t))) \\
 &= \dot{\varphi}_2(\varphi_1(\varphi_1^{-1}(\varphi_2^{-1}(t)))) \dot{\varphi}_1(\varphi_1^{-1}(\varphi_2^{-1}(t))) \\
 &= \dot{\varphi}_2(\varphi_2^{-1}(t)) \dot{\varphi}_1(\varphi_1^{-1}(\varphi_2^{-1}(t))) \\
 &= (\varphi_2^{-1})^* [\dot{\varphi}_2 (\varphi_1^{-1})^* \dot{\varphi}_1],
 \end{aligned}$$

as required.

For $\Psi_1, \Psi_2 \in C^\infty(J_0^1) \otimes \mathbf{C}$ and $\varphi \in \text{Diff}_+ \mathbf{R}$,

$$\begin{aligned}
 (\Xi_\varphi \Psi_1 \mid \Xi_\varphi \Psi_2) &= \int_{J_0^1} \overline{\Xi_\varphi \Psi_1(t, x, \dot{x})} \Xi_\varphi \Psi_2(t, x, \dot{x}) dt d^3 x d^3 \dot{x} \\
 &= \int_{J_0^1} \overline{(\varphi^{-1})^{1*} \{\dot{\varphi}(t) \Psi_1(t, x, \dot{x})\}} (\varphi^{-1})^{1*} \{\dot{\varphi}(t) \Psi_2(t, x, \dot{x})\} dt d^3 x d^3 \dot{x} \\
 &= \int_{J_0^1} [\dot{\varphi}(\varphi^{-1}(t))]^2 \overline{\Psi_1(\varphi^{-1}(t), x, \dot{x}/(\varphi^{-1}) \dot{\varphi}(t))} \{\Psi_2(\varphi^{-1}(t), x, \dot{x}/(\varphi^{-1}) \dot{\varphi}(t))\} dt d^3 x d^3 \dot{x}.
 \end{aligned}$$

Note that the inverse function theorem guarantees

$$\dot{\varphi}(\varphi^{-1}(t)) = \frac{1}{(\varphi^{-1}) \dot{\varphi}(t)}.$$

Introducing new variables

$$\bar{t} = \varphi^{-1}(t), \quad \bar{x} = x \text{ and } \bar{x}' = \dot{x}/(\varphi^{-1}) \dot{\varphi}(t),$$

yields

$$\begin{aligned}
 d\bar{t} &= (\varphi^{-1}) \dot{\varphi}(t) dt, \\
 d\bar{x} &= dx, \\
 d\bar{x}' &= \frac{1}{(\varphi^{-1}) \dot{\varphi}(t)} d\dot{x} - \frac{(\varphi^{-1}) \ddot{\varphi}(t) \dot{x}}{[(\varphi^{-1}) \dot{\varphi}(t)]^2} dt, \\
 d\bar{t} d^3 \bar{x} d^3 \bar{x}' &= \frac{1}{((\varphi^{-1}) \dot{\varphi}(t))^2} dt d^3 x d^3 \dot{x},
 \end{aligned}$$

so that

$$dt d^3 x d^3 \dot{x} = [(\varphi^{-1})'(t)]^2 d\bar{t} d^3 \bar{x} d^3 \bar{x}'.$$

Therefore,

$$\begin{aligned} (\Xi_\varphi \Psi_1 \mid \Xi_\varphi \Psi_2) &= \int_{J_0^1} [\dot{\varphi}(\varphi^{-1}(t))]^2 \overline{\Psi_1(\bar{t}, \bar{x}, \bar{x}')} \Psi_2(\bar{t}, \bar{x}, \bar{x}') dt d^3 x d^3 \dot{x} \\ &= \int_{J_0^1} \frac{1}{[(\varphi^{-1})'(t)]^2} \overline{\Psi_1(\bar{t}, \bar{x}, \bar{x}')} \Psi_2(\bar{t}, \bar{x}, \bar{x}') [(\varphi^{-1})'(t)]^2 d\bar{t} d^3 \bar{x} d^3 \bar{x}' \\ &= \int_{J_0^1} \overline{\Psi_1(\bar{t}, \bar{x}, \bar{x}')} \Psi_2(\bar{t}, \bar{x}, \bar{x}') d\bar{t} d^3 \bar{x} d^3 \bar{x}'. \end{aligned}$$

q.e.d.

Note that, for $\Psi, \Psi' \in C_0^\infty(J_0^1) \otimes \mathbf{C}$, the integral defining the scalar product (116), and

$$(\Psi' \mid \Psi) = \int_{J_0^1} \overline{\Psi'(t, x, \dot{x})} \Psi(t, x, \dot{x}) dt d^3 x d^3 \dot{x} \quad (124)$$

can be interpreted as the evaluation on Ψ of the generalized function (distribution) $\Psi' \in (C_0^\infty(J_0^1) \otimes \mathbf{C})'$. The representation Ξ of $\text{Diff}_+ \mathbf{R}$ on $C_0^\infty(J_0^1) \otimes \mathbf{C}$ extends to a representation of $\text{Diff}_+ \mathbf{R}$ on $(C_0^\infty(J_0^1) \otimes \mathbf{C})'$, which we also denote by Ξ , such

$$(\Xi_\varphi \Psi' \mid \Psi) = (\Psi' \mid \Xi_{\varphi^{-1}} \Psi) \quad (125)$$

for $\varphi \in \text{Diff}_+ \mathbf{R}$, $\Psi' \in (C_0^\infty(J_0^1) \otimes \mathbf{C})'$. Note that, if $\Psi' \in C_0^\infty(J_0^1) \otimes \mathbf{C}$, then the definition of the action Ξ_φ on Ψ' , given here, coincides with the definition given in equation (123).

It remains to examine the space of $\text{Diff}_+ \mathbf{R}$ -invariant functions. Since the group $\text{Diff}_+ \mathbf{R}$ is not compact, the only compactly supported $\text{Diff}_+ \mathbf{R}$ -invariant function in $C_0^\infty(J_0^1) \otimes \mathbf{C}$ is identically zero. Hence, $\text{Diff}_+ \mathbf{R}$ -invariant functions are in $C^\infty(J_0^1) \otimes \mathbf{C}$.

LEMMA 5.3. *For $\mathbf{q} = (t_0, x_0, \dot{x}_0) \in J_0^1$, the orbit $\exp(\text{diff}_+ \mathbf{R})(\mathbf{q})$ of $\text{diff}_+ \mathbf{R}$ through \mathbf{q} coincides with the orbit $\text{Diff}_+ \mathbf{R}(\mathbf{q})$ of $\text{Diff}_+ \mathbf{R}$ through \mathbf{q}*

$$\exp(\text{diff}_+ \mathbf{R})(\mathbf{q}) = \text{Diff}_+ \mathbf{R}(\mathbf{q}).$$

Proof. For $X = \tau \partial_t \in \text{diff}_+ \mathbf{R}$, the integral curves of $X^1 = \tau \frac{\partial}{\partial t} - \dot{\tau} \dot{x} \frac{\partial}{\partial \dot{x}}$ satisfy the differential equations

$$\frac{dt}{ds} = \tau, \quad \frac{dx}{ds} = 0, \quad \text{and} \quad \frac{d\dot{x}}{ds} = -\dot{\tau} \dot{x}.$$

Hence,

$$\frac{dt}{\tau} = ds$$

and

$$\int_{t_0}^t \frac{dt'}{\tau(t')} = s.$$

Choosing $\tau(t) = e^{-t}$,

$$\int_{t_0}^t \frac{dt'}{\tau(t')} = \int_{t_0}^t e^{t'} dt' = e^t - e^{t_0}.$$

Hence, $e^t - e^{t_0} = s$, which implies that

$$t = \log |s + e^{-t_0}|.$$

But the range of the logarithm is $(-\infty, \infty)$. Therefore, the range of values of t on the orbit of $\text{diff}_+ \mathbf{R}$ through q is $(-\infty, \infty)$.

Moreover, for $i = 1, 2, 3$,

$$\frac{d\dot{x}_i}{ds} = -\dot{\tau}\dot{x}_i$$

implies

$$\frac{d\dot{x}_i}{\dot{x}_i} = -\dot{\tau}(t(s)) ds$$

so that

$$\dot{x}(s) = \dot{x}_0 \exp\left(-\int_0^s \dot{\tau}(t(s)) ds\right).$$

Since $\tau(t)$ is an arbitrary function of t , it follows that the orbit of $\text{diff}_+ \mathbf{R}$ through $q = (t_0, x_0, \dot{x}_0)$ is

$$\exp(\text{diff}_+ \mathbf{R})(q) = \{(u, x_0, e^v \dot{x}) \mid (u, v) \in \mathbf{R}^2\}.$$

The action of $\text{Diff}_+ \mathbf{R}$ on J_0^1 is

$$\text{Diff}_+ \mathbf{R} \times J_0^1 \rightarrow J_0^1 : (\varphi, (t, x, \dot{x})) \mapsto \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)}\right),$$

where $\dot{\varphi}(t) > 0$. Since, $\varphi(t)$ and $\dot{\varphi}(t)$ are independent, it follows that the orbit of $\text{Diff}_+ \mathbf{R}$ through q is

$$\text{Diff}_+ \mathbf{R}(q) = \{(u, x_0, w\dot{x}) \mid (u, w) \in \mathbf{R}^2, w > 0\}.$$

Hence, $\exp(\text{diff}_+ \mathbf{R})(q) = \text{Diff}_+ \mathbf{R}(q)$.

q.e.d.

THEOREM 5.4. *A function $\Psi' \in C^\infty(J_0^1) \otimes \mathbf{C}$ is $\text{Diff}_+ \mathbf{R}$ -invariant if and only if*

$$\mathcal{Q}_{J_\tau} \Psi' = 0$$

for all $\tau \in \text{diff}_+ \mathbf{R}$.

Proof. If $\Psi' \in C^\infty(J_0^1) \otimes \mathbf{C}$ is $\text{Diff}_+ \mathbf{R}$ -invariant, then it is invariant under the action of every one-parameter subgroup of $\text{Diff}_+ \mathbf{R}$. By proposition 5.1, actions of one-parameter subgroups of $\text{Diff}_+ \mathbf{R}$ on $C^\infty(J_0^1) \otimes \mathbf{C}$ are generated by $\frac{-i}{\hbar} \mathcal{Q}_{J_\tau}$ for $\tau \in \text{diff}_+ \mathbf{R}$. Hence, $\mathcal{Q}_{J_\tau} \Psi' = 0$ for all $\tau \in \text{diff}_+ \mathbf{R}$.

Conversely, suppose that Ψ' is a function in $C^\infty(J_0^1) \otimes \mathbf{C}$ such that $\mathcal{Q}_{J_\tau} \Psi' = 0$ for all $\tau \in \text{diff}_+ \mathbf{R}$. Hence, $\Xi_{\varphi_\epsilon} \Psi' = \Psi'$ for every one-parameter subgroup φ_ϵ of $\text{Diff}_+ \mathbf{R}$. Recall that, for $\varphi \in \text{diff}_+ \mathbf{R}$,

$$\begin{aligned} \Xi_\varphi \Psi'(t, x, \dot{x}) &= (\varphi^{-1})^{1*} [\dot{\varphi}(t) \Psi'(t, x, \dot{x})] = [(\varphi^{-1})^* \dot{\varphi}(t)] [(\varphi^{-1})^{1*} \Psi'(t, x, \dot{x})] \\ &= \dot{\varphi}(\varphi^{-1}(t)) \Psi'(\varphi^{-1}(t, x, \dot{x})) = \frac{1}{\frac{d\varphi^{-1}(t)}{dt}} \Psi'(\varphi^{-1}(t, x, \dot{x})), \end{aligned}$$

because

$$\varphi \circ \varphi^{-1} = \text{identity}$$

implies

$$\dot{\varphi}(\varphi^{-1}(t)) \frac{d\varphi^{-1}(t)}{dt} = 1.$$

Therefore,

$$\Xi_{\varphi^{-1}} \Psi'(t, x, \dot{x}) = \frac{1}{\dot{\varphi}(t)} \Psi'(\varphi(t, x, \dot{x})).$$

It follows from the lemma above that there exists a finite sequence of one-parameter subgroups $\varphi_{\epsilon_1}, \dots, \varphi_{\epsilon_k}$ such that

$$\varphi(t, x, \dot{x}) = \varphi_{\epsilon_k}(\dots(\varphi_{\epsilon_1}(t, x, \dot{x}))\dots).$$

Since

$$\begin{aligned} \varphi(t, x, \dot{x}) &= \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)} \right), \\ \varphi(t) &= \varphi_{\epsilon_k}(\dots(\varphi_{\epsilon_1}(t))\dots) = \varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1}(t) \end{aligned}$$

and

$$\dot{\varphi}(t) = \dot{\varphi}_{\epsilon_k}(\dots(\varphi_{\epsilon_1}(t))\dots) \dots \dot{\varphi}_{\epsilon_1}(t) = \frac{d}{dt} \varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1}(t).$$

Hence,

$$\begin{aligned} \Xi_{\varphi^{-1}} \Psi'(t, x, \dot{x}) &= \frac{1}{\dot{\varphi}(t)} \Psi'(\varphi(t, x, \dot{x})) = \frac{1}{\dot{\varphi}(t)} \Psi' \left(\varphi(t), x, \frac{\dot{x}}{\dot{\varphi}(t)} \right) \\ &= \frac{1}{\frac{d}{dt} \varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1}(t)} \Psi' \left(\varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1}(t), x, \frac{\dot{x}}{\frac{d}{dt} \varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1}(t)} \right) \\ &= \Xi_{(\varphi_{\epsilon_k} \circ \dots \circ \varphi_{\epsilon_1})^{-1}} \Psi'(t, x, \dot{x}) = \Xi_{\varphi_{\epsilon_1}^{-1} \circ \dots \circ \varphi_{\epsilon_k}^{-1}} \Psi'(t, x, \dot{x}) = \Xi_{\varphi_{\epsilon_1}^{-1} \circ \dots \circ \varphi_{\epsilon_k}^{-1}} \Psi'(t, x, \dot{x}) \\ &= \Xi_{\varphi_{\epsilon_1}^{-1}} \dots \Xi_{\varphi_{\epsilon_k}^{-1}} \Psi'(t, x, \dot{x}) = \Psi'(t, x, \dot{x}), \end{aligned}$$

because Ψ' is invariant under the action of one-parameter subgroups of $\text{Diff}_+ \mathbf{R}$.
q.e.d.

5.3. Quantum implementation of constraints. In the Bleuler-Gupta approach, the extended phase space is quantized first. This associates to each constraint function the corresponding quantum operator. The next step is to implement the constraint conditions on the quantum level. This is done by placing a restriction on the states of the system.

DEFINITION 5.5. *Admissible quantum states* are eigenstates of the quantum operators associated to the constraint functions corresponding to the joint eigenvalue zero.

For elastica, the classical constraints are

$$\begin{aligned} p_t &= 0, \\ \langle p_{\dot{x}}, \dot{x} \rangle &= 0, \\ |\dot{x}|^3 \langle p_{\dot{x}}, p_{\dot{x}} \rangle + 4 \langle p_x, \dot{x} \rangle &= 0. \end{aligned}$$

However, linear combinations of these functions with smooth coefficients leads to further functions that vanish on the range of Legendre transformation. We might not be able to quantize these functions in our chosen quantization scheme. This is why it is important to have criteria that help select a convenient basis of the ideal of functions that vanish on the range of Legendre transformation.

The first two constraint conditions $p_t = 0$ and $\langle p_{\dot{x}}, \dot{x} \rangle = 0$ are equivalent to vanishing of the momenta \mathcal{J}_τ for the action of one-parameter subgroups of $\text{Diff}_+ \mathbf{R}$. Therefore, the quantum implementation of these constraints is the requirement that the admissible wave functions Ψ should satisfy the conditions

$$\mathcal{Q}_{\mathcal{J}_\tau} \Psi = 0 \text{ for } \tau \in \text{diff}_+ \mathbf{R}.$$

By the results of the preceding section, this is equivalent to requiring that admissible states should be invariant under the action of the quantization representation of $C^\infty(J_0^1) \otimes \mathbf{C}$.

The third constraint function has no immediately clear geometric interpretation. We have used the freedom of the choice of generators of the ideal of constraint functions and replaced it by the reparametrization invariant function

$$h = \frac{|\dot{x}|^2}{4} \langle p_{\dot{x}}, p_{\dot{x}} \rangle + \frac{\langle p_x, \dot{x} \rangle}{|\dot{x}|}.$$

The quantum operator corresponding to h is

$$\mathcal{Q}_h = -\frac{\hbar^2}{4} \left(|\dot{x}|^2 \Delta - \frac{1}{3} \right) - \frac{i\hbar}{|\dot{x}|} \left\langle \dot{x}, \frac{\partial}{\partial \dot{x}} \right\rangle.$$

Thus, admissible states Ψ of quantum elastica are also required to satisfy the equation $\mathcal{Q}_h \Psi = 0$.

SUMMARY 5.6. The admissible states of quantum elastica are given by functions $\Psi \in C^\infty(J_0^1) \otimes \mathbf{C}$ that are invariant under the quantization representation of the reparametrization group $\text{Diff}_+ \mathbf{R}$ and satisfy the equation

$$-\frac{\hbar^2}{4} \left(|\dot{x}|^2 \Delta - \frac{1}{3} \right) \Psi - \frac{i\hbar}{|\dot{x}|} \left\langle \dot{x}, \frac{\partial}{\partial \dot{x}} \right\rangle \Psi = 0.$$

As we have mentioned before, admissible functions of quantum elastica are not square integrable on J_0^1 . Therefore, in this formulation of the theory, we need to introduce a new scalar product on the space of admissible states using physical or geometric criteria. For example, we could use $\text{Diff}_+ \mathbf{R}$ invariance of admissible quantum states and relate them to smooth functions on the space S of $\text{Diff}_+ \mathbf{R}$ - orbits in J_0^1 , which satisfy the differential equation obtained from the quantization of h considered as a function on T^*S .

REFERENCES

- [1] K. Bleuler. Eine neue methode sur behandlung der longitudinalen und skalaren photonen. *Helv. Phys. Acta*, 23:567, 1950.
- [2] M. Born. *Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen*. PhD thesis, University of Göttingen, 1906.

- [3] R. Bryant and P. Griffiths. Reduction for constrained variational problems and $\int (k^2/2) ds$. *American journal of mathematics*, 108:525–570, 1986.
- [4] P. Dirac. Hamiltonian methods and quantum mechanics. *Proceedings of the Royal Irish Academy*, 63(3):49–59, 1963.
- [5] L. Euler. *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti*, volume E065. chapter Additamentum I. eulerarchive.org, 1744.
- [6] K. Foltinek. The Hamiltonian theory of elastica. *American Journal of Mathematics*, 116:1479–1488, 1994.
- [7] H. Grässer. *A monograph on the general theory of second order parameter-invariant problems in the calculus of variations*. Number M5 in Mathematical communications of the University of South Africa. Pretoria, 1967.
- [8] S. Gupta. Theory of longitudinal photons in quantum electrodynamics. *Proc. Phys. Soc.*, A63:681, 1950.
- [9] Y. Kosmann-Schwarzbach. *The Noether theorems*. Springer, New York, 2011.
- [10] D. Krupka, O. Krupkova, and D. Saunders. Cartan - Lepage forms in geometric mechanics. *International Journal of Non-Linear Mechanics*, 47:1154–1160, 2012.
- [11] Olga Krupkova. Noether theorem, 90 years on. *AIP Conf.Proc.*, 1130:159–170, 2009.
- [12] J. Langer and D. Singer. Knotted elastic curves in \mathbf{R}^3 . *Journal of the London Mathematical Society*, 30(2):512–520, 1984.
- [13] R. Levien. The elastica: a mathematical history. Technical Report UCB/EECS-2008-103, EECS Department, University of California, Berkeley, Aug 2008.
- [14] R. Levien. *From Spiral to Spline: Optimal Techniques in Interactive Curve Design*. PhD thesis, University of California, Berkeley, 2009.
- [15] J.M. Lévy-Leblond. Conservation laws for gauge-variant Lagrangians in classical mechanics. *American journal of physics*, 39(5):502–506, 1971.
- [16] J. Logan. *Invariant variational principles*. Academic Press, 1977.
- [17] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Mathematisch-physikalische Klasse*, pages 235–258, 1918.
- [18] M. Ostrogradski. Memoires sur les equations differentielles relatives au probleme des isoperimetres. *Mem. Ac. St. Petersburg*, VI:385, 1850.
- [19] J. Śniatycki. *Geometric quantization and quantum mechanics*. Number 30 in Applied mathematical sciences. Springer-Verlag, 1980.
- [20] W. Wunderlich. über ein abwickelbares möbiusband. *Monatshefte für Mathematik*, 66:276–289, 1962.

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